1. A stretch of coastline is inhabited by an animal with (positive) population density $N(X,\, T)$. Its evolution is modelled by the equation

$$N_T = rN - \frac{PN}{N + N_0} + DN_{XX} \ ,$$

where $r,\ P,\ N_0$ and D are positive constants. Explain the biological processes included in this model.

Defining suitable new variables, obtain the non-dimensional equation

$$u_t = u - \frac{pu}{1+u} + u_{xx} \ .$$

Discuss the large-time behaviour of spatially uniform solutions of this equation.

When $p > 1$, investigate travelling wave solutions $u = f(x - ct)$ in the $(f,\, f')$ plane. For what values of c does there exist a solution obeying the boundary conditions

$$u \to 0 \quad \text{as} \quad x \to -\infty, \qquad u \to (p-1) \quad \text{as} \quad x \to \infty \ ?$$

2. Two pigments, with non-dimensional concentrations $u(x,\, y,\, t)$ and $v(x,\, y,\, t)$, occupy a rectangular area Ω. The chemicals react and diffuse according to the equations

$$\left.\begin{aligned} u_t &= 1 + 3u - u^2 - 3v + \nabla^2 u \\ v_t &= 6 + 4u - 9v - v^2 + 20\nabla^2 v \end{aligned}\right\} \quad \text{in} \quad \left\{\begin{aligned} 0 < x < 5\pi \\ 0 < y < 4\pi \end{aligned}\right. ,$$

together with the no-flux boundary conditions on $\partial\Omega$

$$\begin{pmatrix} u \\ v \end{pmatrix}_x = 0 \quad \text{on} \quad x = 0,\, 5\pi \quad \text{and} \quad \begin{pmatrix} u \\ v \end{pmatrix}_y = 0 \quad \text{on} \quad y = 0,\, 4\pi \, .$$

Investigate the stability of the uniform state $u = v = 1$, by writing

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} e^{\lambda t} \phi(x, y) \, ,$$

where $0 < \varepsilon \ll 1$ while u_0, v_0 and the growth rate λ are unknown constants. Formulate and solve the problem for the eigenfunctions $\phi(x, y)$ and associated eigenvalues k^2.

Find an equation for λ in terms of k^2, and deduce that the uniform state is unstable only if $0.2 < k^2 < 0.25$.
Hence show that only one mode ϕ has a strictly positive growth rate.

Sketch a plausible pattern to emerge from this instability, stating clearly what criterion you are using.

3. The functions $u(t)$ and $v(t)$ obey the equations

$$u' = \varepsilon(v - u) \qquad \text{and} \qquad v' = f(v) - u \ ,$$

where the positive constant ε and the function f are known. State the conditions for $(u,\, v) = (u_0,\, v_0)$ to be a stable, steady state.

In the FitzHugh-Nagumo model of neurone behaviour,

$$f(v) = I - v(v - v_1)(v - v_2) \qquad \text{where} \quad v_2 > v_1 > 0 \ .$$

$v(t)$ is the electric potential, while I, v_1 and v_2 are known constants.

(a) When $I = 0$, determine the condition for there to exist two stable, steady states.

(b) Assume that only one steady state exists for $I = 0$ and $\varepsilon \ll 1$. Sketch the two trajectories in the $(v,\, u)$ plane corresponding to $u(0) = 0$ and $v(0) = v_1 \pm \delta$, for some small δ. You should include the curve $u = f(v)$ on your figure.

Plot the potential $v(t)$ for each trajectory, and explain the biological significance of the difference in behaviour.

(c) An experiment is performed with $I > 0$. Show graphically that it is possible for the system to have a unique equilibrium which is unstable. Sketch a plausible trajectory in this case.

4. Blood contains dissolved oxygen, O_2, haemoglobin, Hb, and oxygenated haemoglobin, HbO_2, with concentrations u, v and h respectively. The total haemoglobin remains constant, $v+h=v_0$, during the reversible reaction

$$O_2 + Hb \underset{k_-}{\overset{k_+}{\rightleftarrows}} HbO_2 \ .$$

Find the equilibrium concentration of h as a function of u and sketch it roughly.

Blood flows into the alveoli of the lungs with constant velocity V where, for $x > 0$, it comes into contact with a high O_2 concentration. The oxygen diffuses with coefficient D, much faster than do the larger haemoglobin molecules. The oxygen concentration $u(x,\, y)$ therefore satisfies

$$V[u + h(u)]_x = Du_{yy} \qquad \text{in} \quad x > 0,\ y > 0\ , \qquad \text{with}$$

$$u(x,0) = u_0, \qquad u(0,\, y) = u_\infty \qquad \text{and} \quad u \to u_\infty \qquad \text{as} \quad y \to \infty\ .$$

Approximating $h'(u) \simeq M$, a constant, seek a solution of the form

$$u(x,\, y) = f(\eta) \qquad \text{where} \quad \eta = \frac{y}{2\sqrt{\nu x}} \qquad \text{and} \quad \nu = \frac{D}{(1+M)V}\ .$$

Deduce an ODE with boundary conditions for $f(\eta)$ and solve it.
Evaluate the oxygen absorption rate, $-Du_y(x,0)$.

5. Some fish are almost spherical when small, but become long and thin as they grow bigger. Suggest a plausible reason for this.

At $t = 0$, a fish is at rest at $X = 0$. It swims for a time T, exerting a unit force F and then glides to rest. Its velocity $V(t)$ is modelled by the equation

$$\frac{dV}{dt} = F(t) - \frac{2V}{R} - V^2 \qquad \text{where} \quad F(t) = \begin{cases} 1 & \text{for} \quad 0 < t < T \\ 0 & \text{for} \quad T < t\ . \end{cases}$$

What might the (constant) parameter R represent?

Writing $k = 1/R$, $u(t) = V(t) + k$ and $c = \sqrt{1+k^2}$, show that

$$u = \begin{cases} c \tanh[ct + A] & \text{for} \quad 0 < t < T \\ k \coth[kt + B] & \text{for} \quad T < t\ , \end{cases}$$

where $\sinh A = k$. Give an equation determining B.

Defining $X = \int_0^t V\,dt$, show that

$$X = \ln\left[\tfrac{1}{c}\big(\cosh[ct + A]\big)\right] - kt \qquad \text{for} \quad 0 < t < T\ ,$$

$$\text{and that} \qquad X \to L = \ln\left[\frac{\cosh[cT + A]}{2c\sinh[kT + B]}\right] + B \qquad \text{as } t \to \infty.$$

What is the total work done by the fish in travelling this distance?