M_4^3A21 Handout $2\frac{1}{2}$: The oxygenation of blood

In the alveoli of the lungs, oxygen diffuses from the air into the blood stream where it reacts reversibly and quickly with haemoglobin. The haemoglobin diffuse much more slowly than oxygen, so that the equation describing the oxygen transport is

$$\left.\begin{array}{l} \big(1+M(u)\big)u_t = Du_{xx} \qquad \text{in} \quad x>0, \quad t>0 \\ \text{with} \quad u(0,t)=0\ , \qquad u(\infty,t)=1 \qquad \text{and} \quad u(x,0)=1\ . \end{array}\right\} \tag{6.8}$$

The oxygen and haemoglobin concentrations are $s(x,\, t)$ and $c(x,\, t)$, while

$$u = \frac{s-s_0}{s_\infty - s_0}\ , \qquad h = \frac{c-c_0}{s_\infty - s_0} \qquad \text{and} \quad M = \frac{dc}{ds} = \frac{dh}{du}\ .$$

The air/blood boundary is at $x=0$, where $s=s_0$, $c=c_0$; at some distance the oxygen level is $s=s_\infty$, while the function $c(s)$ is determined by chemical kinetics (6.7).

If we make the transformation $x=\lambda X$, $t=\lambda^2 T$, $u(x,\, t) = U(X,\, T)$, where λ is an arbitrary positive constant, we find that $U(X,\, T)$ satisfies exactly the same equation and boundary conditions as $u(x,\, t)$. Assuming this problem has a unique solution, we can infer that

$$u(x,\, t) = u(x/\lambda,\, t/\lambda^2) \qquad \text{for any } \lambda > 0.$$

This strongly suggests that u only depends on x and t only in the form x^2/t, so that we seek a **similarity solution:**

$$u(x,\, t) = f(\zeta) \qquad \text{where} \quad \zeta = \frac{x}{(4Dt)^{1/2}}\ .$$

The factor of $4D$ is included for simplicity later. We then have

$$u_t = f'(\zeta)\left(\frac{-x}{4D^{1/2}t^{3/2}}\right) \qquad \text{and} \quad u_{xx} = \frac{f''(\zeta)}{4Dt}\ .$$

so that (6.8) reduces to

$$f'' + 2\big(1+M(f)\big)\zeta f' = 0 \qquad \text{in} \quad \zeta>0 \qquad \text{with} \quad f(0)=0 \qquad \text{and} \quad f(\infty)=1\ . \tag{6.9}$$

Note that this is an ODE for $f(\zeta)$. Had x or t remained in the equation explicitly the similarity solution would not work. Also, we see that the boundary conditions at $t=0$ and $x\to\infty$ are compatible with the proposed solution.

Equation (6.9) can be solved numerically for any functional form of $M(f)$. If we approximate the curve $c=c(s)$ by a straight line so that $M \simeq M_0 = h_\infty$, a constant, then (6.9) can be integrated to give

$$\ln f' + (1+M_0)\zeta^2 = \text{constant}$$

or using the boundary conditions

$$f = A\int_0^\zeta e^{-(1+M_0)z^2}\,dz = \operatorname{erf}\left(\zeta\sqrt{1+M_0}\right)$$

where the error function $\operatorname{erf}(z)$ is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt \qquad \text{so that} \quad \operatorname{erf}(\infty) = 1 \ .$$

Thus in terms of x and t,

$$u(x,\, t) \simeq \operatorname{erf}\left[\tfrac{1}{2}x\sqrt{\frac{1+M_0}{Dt}}\right] \ .$$

The rate of oxygen absorption into the blood, $N(t)$, is the diffusive flux at $x = 0$, so in this model

$$N = N_1 \equiv -D\frac{\partial s}{\partial x}\Big|_{x=0} = D(s_0 - s_\infty)\frac{\partial u}{\partial x}\Big|_{x=0} = (s_0 - s_\infty)\sqrt{\frac{D}{\pi t}}\sqrt{1+M_0} \ . \tag{6.10}$$

From the shape of $c(s)$ we know that in reality $c > M_0 s$. Physically, we expect the presence of more haemoglobin to hasten the absorption of oxygen, and so (6.10) should be an underestimate of the real absorption rate. An overestimate can be obtained by formally replacing $M(f)$ by a delta-function, so that

$$h = 0 \qquad \text{for} \quad 0 < u < 1 \qquad \text{and} \quad h = h_\infty \qquad \text{for} \quad u > 1.$$

This assumes a two layer structure with a discontinuity in the haemoglobin concentration at some unknown position $\zeta = \zeta_0$. While $M = 0$, the above solution applies and we have

$$f = \operatorname{erf}(\zeta)/\operatorname{erf}(\zeta_0) \qquad \text{for} \quad 0 < \zeta < \zeta_0 \qquad \text{and} \quad f = 1 \qquad \text{for} \quad \zeta > \zeta_0 \ .$$

At $\zeta = \zeta_0$ both $f'(\zeta)$ and h are discontinuous, and for consistency of the model the oxygen flux must balance. Integrating (6.9) over the discontinuity gives, for arbitrarily small ε,

$$\Big[f'\Big]_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} + 2\zeta_0\Big[h\Big]_{\zeta_0-\varepsilon}^{\zeta_0+\varepsilon} = O(\varepsilon) \ .$$

This determines the necessary value of ζ_0, which, for given M_0, must satisfy

$$2M_0\zeta_0 e^{\zeta_0^2}\int_0^{\zeta_0} e^{-z^2}\,dz = 1 \ .$$

For this second approximation the oxygen absorption rate is given by

$$N = N_2 \equiv D(s_0 - s_\infty)\frac{\partial u}{\partial x}\Big|_{x=0} = (s_0 - s_\infty)\sqrt{\frac{D}{\pi t}}\,\frac{1}{\operatorname{erf}(\zeta_0)} \ . \tag{6.11}$$

The real oxygen uptake rate, N, which can be found numerically, must satisfy $N_1 < N < N_2$. In fact, it can be shown that $N_2/N_1 < \sqrt{\pi/2}$ so that the approximations are reasonable.