M^3_4A21 Mathematical Biology: Example sheet 1 **A.J. Mestel**

Population Dynamics of a single species; Fisher's equation.

1. An organism, with population density $U(X,\,T)$, has a logistic growth and diffuses in space, so that

$$U_T = RU(1-\frac{U}{Q}) + DU_{XX} \qquad \text{in} \quad a < X < b\ ,$$

where R, Q and D are known positive constants.
Defining x, t and $u(x,\,t)$ appropriately, obtain a non-dimensional version of this equation

$$u_t = u(1-u) + u_{xx} \qquad \text{in} \quad 0 < x < L\ , \tag{1}$$

and give a formula for L.

2. The population density $u(x,t)$ obeys the Fisher equation (1) together with the boundary conditions

$$u(0,\,t) = 0 = u(L,\,t) \qquad \text{and} \quad u(x,\,0) = u_0(x) \geqslant 0\ .$$

You may assume that $u(x,\,t) \geqslant 0$ for all x and t.
Defining a sort of "energy," $E(t) = \frac{1}{2}\int_0^L u^2\,dx$, show that

$$\frac{dE}{dt} = \int_0^L u^2(1-u)\,dx - \int_0^L u_x^2\,dx\ .$$

Assuming the Poincaré inequality for smooth functions $\phi(x)$ satisfying $\phi(0) = \phi(L) = 0$,

$$\int_0^L \phi_x^2\,dx \geqslant \frac{\pi^2}{L^2}\int_0^L \phi^2\,dx\ , \tag{Poincaré}$$

deduce that the population will die out if $L < \pi$. Write down the corresponding result in terms of the dimensional quantities a, b, R, D and Q of question 1.

3. Suppose, instead, that $u(x,t)$ is subject to the 'no flux' boundary conditions

$$u_x(0,\,t) = 0 = u_x(L,\,t) \qquad \text{and} \quad u(x,0) = u_0(x)\ .$$

This time consider $F(t) = \frac{1}{2}\int_0^L u_x^2\,dx$. Show that if $L < \pi$, then $F \to 0$ as $t \to \infty$, and hence u must tend to a constant for any initial state $u_0(x)$.
[Hint: First differentiate (1) with respect to x, and continue as in the last question.]

4. A swimming pool occupies $0 < x < a$, $0 < y < b$, $0 < z < c$. The density of algae within it, $u(x,\,y,\,z,\,t)$, is assumed to obey the equation and boundary conditions

$$u_t = u(1-u) + \nabla^2 u \qquad \text{with} \quad u = 0 \qquad \text{on} \quad x = 0, a, \quad y = 0, b, \quad z = 0, c\ .$$

Discuss whether or not the 'clean' state $u \equiv 0$ is stable.
Is this a good model, do you think?