M_4^3A21 Mathematical Biology: Example sheet 2 **A.J. Mestel**

Chemical reaction kinetics and Turing pattern formation.

1. A chemical reaction involves the four molecular types A, B, X and Y. The concentrations of A and B are maintained at constant levels A_0 and B_0 by fast external processes, while X and Y are produced by the three reactions

$$A \underset{k_-}{\overset{k_1}{\rightleftarrows}} X \ , \qquad B \overset{k_2}{\longrightarrow} Y \ , \qquad 2X + Y \overset{k_3}{\longrightarrow} 3X \ .$$

Obtain evolution equations for the concentrations $x(t)$ and $y(t)$ of X and Y.
Show that non-dimensionalisation of the form $\tau = k_- t$, $u(\tau) = \omega x$ and $v(\tau) = \omega y$, where $\omega^2 = k_3/k_-$, produces the system

$$u' = a - u + u^2 v \ , \qquad v' = b - u^2 v \ , \tag{1}$$

and state the values of a and b (which are of course positive.)

2. Demonstrate that (1) has a unique stationary point, and find the condition for it to be stable. What is the largest possible value of a for which this equilibrium can be unstable? If it is unstable, what sort of behaviour in the (u, v)-plane do you expect?

3. Discuss whether or not the system

$$u_t = (a - u + u^2 v) + \nabla^2 u \qquad v_t = (b - u^2 v) + d\nabla^2 v$$

can exhibit Turing-like spatial instabilities, identifying any critical relationships.

4. The surface of a zebra's leg is modelled as a cone of angle α, so that in terms of spherical polar coordinates $(r,\ \theta,\ \eta)$ it occupies $\theta = \alpha$, $0 < r < L$, $0 < \eta < 2\pi$. In these coordinates the Helmholtz equation for $\phi(r,\ \eta)$ is

$$\frac{1}{r^2 \sin^2 \alpha} \frac{\partial^2 \phi}{\partial \eta^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + k^2 \phi = 0 \qquad \text{with} \quad \frac{\partial \phi}{\partial r}(L, \eta) = 0 \ .$$

Find the η-independent eigenfunctions and show that the eigenvalues satisfy $\tan kL = kL$. (Hint: write $\phi = \psi(r)/r$.) Roughly how large must k^2 be for (a) striped and (b) non-striped leg patterns to occur?

5. In lectures we considered the following model for testosterone production in males:

$$R' = f(T) - b_1 R \ , \qquad L' = R - b_2 L \ , \qquad T' = L - b_3 T \qquad \text{with} \quad b_i > 0 \ .$$

Suppose the negative feedback function $f(T) = 1/(1 + T^m)$ for some $m > 0$. Show that there is a unique equilibrium which is stable if $m \leqslant 8$. Hints:
(i) The roots of $x^3 + ax^2 + bx + c = 0$ have negative real parts iff $a > 0$, $c > 0$ and $ab > c$.
(ii) For any three positive numbers $\frac{1}{3}(p + q + r) \geqslant (pqr)^{1/3}$.
[It follows that for $m \leqslant 8$ no oscillations in hormone levels are possible. If $m > 8$, however, limit cycles can occur.)