M^3_4A21 Mathematical Biology: Example sheet 3 **A.J. Mestel**

Epidemiology and transport processes.

1. Consider the following model of the spread of AIDS. Let w be the number of people who have developed AIDS, x the number known to be HIV$^+$ who have not yet developed AIDS, y the number of HIV$^+$ unknown to themselves, and z the number of non-infected people.

$$\frac{dw}{dt} = \mu(x+y) - pw \qquad \frac{dx}{dt} = ky - \mu x$$
$$\frac{dy}{dt} = -(k+\mu)y + \lambda yz \qquad \frac{dz}{dt} = -\lambda yz$$

Interpret the positive constants μ, p, k, and λ and explain any assumptions of the model. Initially, $w = w_0$, $x = x_0$, $y = y_0$ and $z = z_0$ and these values are such that w, x and y are initially increasing, while z is decreasing. Describe qualitatively what will happen to the population and give rough sketches of plausible behaviour of y and z against t.
Show that the steady state $(w, x, y, z) = (0, 0, 0, z_1)$ is unstable if $z_1 > \varepsilon$ where $\varepsilon = (k+\mu)/\lambda$. Obtain an equation relating y and z and hence show that

$$y = y_0 + z_0 - z + \varepsilon \log \frac{z}{z_0}\ .$$

Deduce that if ε is small then the final size of the population will be given approximately by

$$z = z_0 \exp[-(y_0+z_0)/\varepsilon]\ .$$

What attributes of the disease make it dangerous to humanity?

2. Oxygen with density $u(x)$ travels through a membrane at $x = 0$ by means of both diffusion and advection. It is absorbed at a rate μ, so that in a steady state

$$Du_{xx} - Vu_x - \mu u = 0 \qquad \text{in} \quad x > 0 \qquad u(0) = u_0, \quad u \to 0 \qquad \text{as} \quad x \to \infty.$$

Find $u(x)$, and compare typical oxygen penetration distances in the limits $V \to 0$, $V \to \infty$ and $D \to 0$.

3. **The return of the spotted cow:** The larvae of a parasitic worm are deposited when a herbivore defecates at $x = y = 0$ at time $t = 0$. The larvae spread out randomly, are eaten at a rate μ, while some of them lay eggs at a rate λ on blades of grass. The densities of larvae, $u(r,\, t)$ and eggs, $E(r,\, t)$ are thus governed by the equations

$$u_t = D\nabla^2 u - \mu u\ , \qquad E_t = \lambda u \qquad \text{with} \quad E(r,\,0) = 0, \quad u(r,\,0) = \delta(\mathbf{r})$$

Writing $u = v(r,t)e^{-\mu t}$, find v (see lecture notes) and hence E. Deduce that as $t \to \infty$ there is a residual density of eggs surrounding the faeces given by

$$E_\infty(r) = \frac{\lambda}{4\pi D}\int_0^\infty \exp\left(-\frac{r^2}{4Dt} - \mu t\right)\frac{dt}{t} = \frac{\lambda}{4\pi D}\int_0^\infty \exp\left[-r\sqrt{\frac{\mu}{4D}}\left(\frac{1}{T} + T\right)\right]\frac{dT}{T}\ .$$