M^3_4A21 Mathematical Biology: Example sheet 4 **A.J. Mestel**

Shear-Dispersion and Biomechanics

Those taking the 4 year degree (M4A21) should submit the solution to question (1) by Friday 14th January at the latest.

1. Flow of air or blood in a cleft $0 < y < a$, can be modelled by a steady Poiseuille velocity in the x-direction,

$$V(y) = \beta y(a-y) \qquad \text{for some constant } \beta.$$

Examine the shear dispersion of a chemical with concentration $u(x, y, t)$ satisfying

$$u_t + V(y)u_x = Du_{xx} + Du_{yy} \qquad \text{with} \quad u_y = 0 \qquad \text{on} \quad y = 0, a\ .$$

Defining y-averages over the cleft cross-section for some quantity w by

$$\overline{w}(x,\, t) = \frac{1}{a}\int_0^a w(x,\, y,\, t)\, dy\ ,$$

find D_{eff}, the effective diffusion coefficient in the x-direction of $\overline{u}$, in terms of D, $\overline{V}$ and a. For what value of D is it minimum?

Find $\overline{u}(x, t)$ for large times explicitly, given the initial condition

$$\overline{u}(x, 0) = e^{-x^2}\ .$$

[Hint: What would a delta-function distribution at some suitable negative time look like at $t = 0$?]

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2. A bird glides down from a great height at constant velocity. Find the angle it makes to the horizontal in terms of its drag and lift coefficients. [You may neglect the Induced Drag, if you wish.] Hence estimate the height to which a soaring bird should climb in a thermal in terms of the separation distance between thermals.

The same bird reaches a thermal, a cylindrical column of air of radius r_t rising with speed v_t relative to the stationary air outside. It glides down in a helix of radius r_0 relative to the thermal. Continue the discussion from lectures to show that it can remain at the same height relative to the ground if

$$r_0^2 = \frac{(v_0^2 - v_t^2)^2}{(\alpha^2 + \beta^2)v_0^4 - g^2} \qquad \text{where} \quad v_0^3 = gv_t/\alpha$$

and the magnitudes of the drag and lift forces at speed v are defined to be $|D| = m\alpha|v|^2$ and $|G| = m\beta|v|^2$.
Infer that it can gain height if $r_t > r_0$.

3. A fish of length L and mass m swims at a Reynolds number only slightly bigger than one, so that it experiences a drag with terms both linear and quadratic in its speed. When it exerts a constant swimming force F, its speed $V(t)$ is therefore modelled by

$$m\frac{dV}{dt} = F - \alpha\mu LV - \beta\rho L^2V^2 \ ,$$

where ρ and μ are the density and viscosity of the water and α and β are constants. Obtain a non-dimensional form of this equation,

$$\frac{dV}{dt} = 1 - \frac{V}{R} - V^2 \qquad \text{and define the (Reynolds number) } R \quad .$$

Show that the work done in travelling a fixed distance X is the same for all values of R.

When this fish is small it is observed to be almost spherical, but as it grows up it becomes long and thin. Suggest a reason for this.

4. Gas flow in the lungs is modelled assuming a periodic pressure gradient in a cleft, so that the uni-directional flow $v(y,t)$ satisfies

$$v_t = \cos\omega t + \nu v_{yy} \qquad \text{in} \quad -a < y < a \qquad \text{with} \quad v(\pm a, t) = 0 \ .$$

Writing $v(y,\, t) = \Re e\left[u(y)e^{i\omega t}\right]$, where $\Re e$ denotes the real part, show that

$$u = \frac{1}{i\omega}\left(1 - \frac{\cosh\left[(1+i)y/\delta\right]}{\cosh\left[(1+i)a/\delta\right]}\right)$$

where

$$\delta = \sqrt{\frac{2\nu}{\omega}} \ .$$

Obtain asymptotic forms of this result when (a) $\delta \gg a$ and (b) $\delta \ll a$.

Sketch the flow profiles in these limits and discuss the (shear-enhanced) downstream diffusion in each case.

[You should find that in one case the shear is very strong near the boundaries but zero in the middle of the cleft. Thus the mixing is very good near the walls but rather weak elsewhere. This model is also relevant to pulsatile blood flow.]