UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2002

This paper is also taken for the relevant examination for the Associateship

M4A33 Industrial Mathematics

DATE: Wednesday, 31 May 2002 TIME: 2 pm –4 pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider thin layer surfactant driven fluid flow along a horizontal impenetrable substrate. You are given that a lubrication-like approximation leads to the governing equations

$$p_z = 0,\ p_x = u_{zz},\ u_x + w_z = 0,$$

$$\Gamma_t + (u_s\Gamma)_x = \Gamma_{xx}/Pe$$

in $|x| < \infty$, $0 < z < h(x,t)$. The fluid pressure is p and the horizontal and vertical fluid velocities are u and w respectively. The surfactant concentration is $\Gamma(x,t)$ and u_s is the horizontal velocity at the surface, with Pe as the constant Peclet number. The boundary conditions are

$$h_t + uh_x = w,\ p = 0,\ \tau_{xz} = u_z = \sigma_x = -\Gamma_x$$

on $z = h(x,t)$ and

$$u = w = 0$$

on $z = 0$. The surface tension σ is related to the surfactant concentration via a linear equation of state $\sigma(\Gamma) = 1 - \Gamma$. Show that the evolution equations for $\Gamma(x,t)$ and $h(x,t)$ are

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{h^2}{2}\frac{\partial \Gamma}{\partial x}\right) = 0, \quad \frac{\partial \Gamma}{\partial t} - \frac{\partial}{\partial x}\left(\Gamma h \frac{\partial \Gamma}{\partial x}\right) = \frac{1}{Pe}\frac{\partial^2 \Gamma}{\partial x^2}.$$

If we assume that the Peclet number, Pe, is $\gg 1$ and the quantity of surfactant placed upon the film surface is proportional to t^α (t is time) use similarity variables

$$\xi = x/t^a,\ h(x,t) = H(\xi),\ \Gamma(x,t) = G(\xi)/t^b$$

to show that the long-time position of the surfactant front, X_{max}, advances such that

$$X_{max} \sim t^{(1+\alpha)/3}.$$

2. Consider the interface, $x = s(t)$, between a fluid in $x < s(t)$ and a solid in $x > s(t)$, if the solid is melting and Latent heat, L, per unit mass is supplied, show that the interface boundary condition is:

$$-K_1\frac{\partial T_1}{\partial x} + K_2\frac{\partial T_2}{\partial x} = L\rho\frac{ds}{dt}.$$

The variables with subscripts $1, 2$ are in the fluid and solid respectively. Both materials have density ρ, the heat conductivities are $K_{1,2}$. The temperatures in the fluid and solid are T_1 and T_2 respectively. Now consider the problem of melting a half space where

$$\frac{\partial T_1}{\partial t} = \kappa_1\frac{\partial^2 T_1}{\partial x^2}, \quad 0 < x < s(t)$$

$$\frac{\partial T_2}{\partial t} = \kappa_2\frac{\partial^2 T_2}{\partial x^2}, \quad s(t) < x < \infty$$

with boundary conditions $T_1 = \Theta_1$ at $x = 0$ and $T_2 = \Theta_2$ as $x \to \infty$. On $x = s(t)$ the interface boundary condition

$$-K_1\frac{\partial T_1}{\partial x} + K_2\frac{\partial T_2}{\partial x} = L\rho\frac{ds}{dt},$$

holds together with $T_1 = T_2 = 0$ on $x = s(t)$. Show that the solution is given by

$$T_1 = \Theta_1 + A\text{erf}\left(\frac{x}{2(\kappa_1 t)^{\frac{1}{2}}}\right) \quad \text{for } 0 < x < s(t)$$

$$T_2 = -\Theta_2 + B\text{erfc}\left(\frac{x}{2(\kappa_2 t)^{\frac{1}{2}}}\right) \quad \text{for } s(t) < x < \infty.$$

A and B are constants to be determined. Show that $s(t) = \alpha t^{\frac{1}{2}}$ where α is a constant that is given by the root of the transcendental relation

$$\frac{K_1\Theta_1 e^{-\alpha^2/4\kappa_1}}{(\pi\kappa_1)^{\frac{1}{2}}\text{erf}\left(\alpha/2\kappa_1^{\frac{1}{2}}\right)} - \frac{K_2\Theta_2 e^{-\alpha^2/4\kappa_2}}{(\pi\kappa_2)^{\frac{1}{2}}\text{erfc}\left(\alpha/2\kappa_2^{\frac{1}{2}}\right)} = \frac{\rho L\alpha}{2}.$$

You can use the definitions

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\tau^2}d\tau, \qquad \text{erfc}(z) = 1 - \text{erf}(z).$$

Turn over...

3. Write an essay on any topic covered in the course but not examined on this paper. Discuss, as appropriate, the industrial significance of the problem, the assumptions of the model and the solution.

N.B. The following formulae may be of use in questions 4 and 5. In terms of spherical polar coordinates $(r,\,\theta,\,\eta)$, an axisymmetric scalar function $u(r,\,\theta)$ and vector $\mathbf{F}(r,\theta) = (F_r,\,F_\theta,\,F_\eta)$ satisfy

$$\nabla u = \left(\frac{\partial u}{\partial r},\,\frac{1}{r}\frac{\partial u}{\partial \theta},\,0\right)$$

$$\nabla \cdot (F_r,\,F_\theta,\,F_\eta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta F_\theta\right) \; .$$

$$\nabla \wedge (F_r,\,F_\theta,\,F_\eta) = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\eta \sin\theta),\; -\frac{1}{r}\frac{\partial}{\partial r}(F_\eta r),\; \frac{1}{r}\frac{\partial (rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right]$$

Laplace's equation $\nabla^2 u = 0$ has the solution

$$u = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta) \; ,$$

where the Legendre function $y = P_n(\cos\theta)$ satisfies

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial y}{\partial \theta}\right) + n(n+1)y = 0 \; .$$

4. A spherical drop of electrically conducting fluid occupies $0 < r < a$ in terms of spherical polar coordinates $(r,\ \theta,\ \eta)$. Its surface is perturbed to the position $\zeta = a$, where $\zeta = r - a\delta P_n(\cos\theta)$. Here P_n is a Legendre function of integer order n, while δ is constant in space with $0 < \delta \ll 1$.

Show that the curvature $K \equiv \nabla \cdot \widehat{\mathbf{n}}$, where $\widehat{\mathbf{n}}$ is the unit normal parallel to $\nabla\zeta$, is given by

$$K = \frac{2}{a} - \frac{\delta}{a} P_n[2 - n(n+1)] + O(\delta^2) \ .$$

The drop is known to carry an electric charge Q, and its surface is at constant electric potential. Find the perturbed electric potential ϕ outside the drop.

Evaluate the normal outward electric stress $\tau = \frac{1}{2}\varepsilon_0(\widehat{\mathbf{n}} \cdot \nabla\phi)^2$ and show that

$$\tau = \tau_0 + \delta \frac{Q^2}{16\pi^2\varepsilon_0 a^4}(n-1)P_n(\cos\theta) + O(\delta^2) \ ,$$

where τ_0 is constant. Deduce the value of p_s, the fluid pressure perturbation on the drop surface, in terms of the surface tension γ and the other parameters.

Given that the hydrodynamics require that if $p_s = \delta A P_n$ then for stability $A > 0$, deduce that for the spherical drop to be stable

$$Q^2 < 64\pi^2\varepsilon_0\gamma a^3 \ .$$

5. In the TIG welding process, a constant electric current I enters a horizontal pool of liquid metal at the origin. In terms of spherical polar coordinates, the metal is assumed to occupy the infinite half-space, $0 < \theta < \frac{\pi}{2}$. The metal has kinematic viscosity ν and density ρ. The current spreads out radially in $0 < \theta < \frac{1}{2}\pi$ with a current density $\mathbf{j} = (j(r),\, 0,\, 0)$, giving rise to the azimuthal magnetic field $\mathbf{B} = (0,\, 0,\, B(r,\theta))$.

Find the functions B and j, and show that

$$\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) = \frac{\mu_0 I^2}{2\pi^2}\left(\frac{1-\cos\theta}{r^4 \sin\theta}\right) .$$

Seek a similarity solution for the resultant motion of the metal given by the streamfunction $\psi = \nu r f(\theta) \equiv \nu r g(\mu)$, where $\mu = \cos\theta$, and

$$\mathbf{u} = \nabla \wedge \left(0,\, 0,\, \frac{\psi}{r\sin\theta}\right) \qquad \text{with} \qquad \underline{\omega} = \nabla \wedge \mathbf{u}\ .$$

Show that

$$\nabla \wedge (\mathbf{u} \wedge \underline{\omega}) = \frac{\nu^2 \sin\theta}{r^4}\,(0,\, 0,\, 2g'g'' + (gg'')')\ ,$$

where $'$ denotes differentiation with respect to μ. Write down the azimuthal component of the vorticity equation given that

$$\nabla^2 \underline{\omega} = \left(0,\, 0,\, \frac{\nu\sin\theta}{r^4}\left[4\mu g''' - (1-\mu^2)g''''\right]\right) .$$

and integrate it three times to show that, for arbitrary constants A, C and D,

$$\tfrac{1}{2}g^2 - (1-\mu^2)g' - 2\mu g + \tfrac{1}{2}K(1+\mu)^2\ln(1+\mu) = A\mu^2 + C\mu + D$$

$$\text{where} \quad K = \frac{\mu_0 I^2}{2\pi^2\nu^2\rho}\ .$$

Numerical integration of this last equation with suitable boundary conditions shows that on the axis $\theta = 0$ the velocity is infinite if K is large enough. How might a real fluid avoid this singularity?

