UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2003
This paper is also taken for the relevant examination for the Associateship

M4A33 Industrial Mathematics

DATE: Wednesday, 31 May 2003 TIME: 2.00 pm–4.00 pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Let us consider the melting of ice by hot water. We shall idealize the situation by considering a semi-infinite region of ice in $x > s(t)$ and have the hot water in $x < s(t)$, where $s(t)$ is the time-dependent position of the interface between ice and water.

By considering a small element δx of ice at the interface that melts in a time δt show that the boundary condition that is imposed at the interface is that

$$-k_1 \frac{\partial T}{\partial x}\Big|_{x=s-} + k_2 \frac{\partial T}{\partial x}\Big|_{x=s+} = \rho L \frac{ds}{dt}$$

where L is Latent heat per unit mass required to melt the ice, $x = s-$ lies just to the left of the interface and $x = s_+$ is to the right. The parameters k_1 and k_2 are the conductivities of the water and ice respectively.

Now we consider the simpler situation when $k_1 = k$ and $k_2 = 0$. Let us assume that the water is constantly refreshed at the interface and is at a constant temperature $T = T_m > T_0$ at the interface. T_0 is the constant temperature of the ice at infinity.

We are required to solve the heat equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

in $x > s(t)$ with boundary conditions

$$T \to T_0 \quad \text{as} \quad x \to \infty$$

and

$$T = T_m, \quad -k\frac{\partial T}{\partial x} = \rho L \frac{ds}{dt}$$

on $x = s(t)$. Given that $\mathrm{erfc}[x/2(\alpha t)^{1/2}]$ is a solution of the heat equation,

[The following integral may be useful

$$\mathrm{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-p^2}\, dp$$

]

show that the solution that satisfies the boundary conditions is given by

$$T = T_0 + \frac{T_m - T_0}{\mathrm{erfc}(\lambda)} \mathrm{erfc}(x/2(\alpha t)^{1/2})$$

where λ is a constant defined as $\lambda = s(t)/2(\alpha t)^{1/2}$. Show that it is determined from

$$\frac{k}{\alpha\sqrt{\pi}}\frac{(T_m - T_0)}{\mathrm{erfc}(\lambda)}e^{-\lambda^2} = \rho L \lambda.$$

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2. Flow is driven in a slightly curved, circular pipe by an oscillating pressure gradient. In terms of cylindrical polar coordinates $(r,\,\theta,\,z)$, with z pointing along the pipe axis, the pipe surface is $r = 1$ and the velocity is written

$$\mathbf{u} = \left(\tfrac{1}{r}\psi_\theta,\, -\psi_r,\, w\right) \qquad \text{and the vorticity} \quad \Omega = -\nabla^2\psi\ .$$

For constant K and p, the unsteady Dean equations for this flow are

$$\begin{aligned} w_t + \tfrac{K}{r}(\psi_\theta w_r - \psi_r w_\theta) &= \cos(2p^2t) + \nabla^2 w \\ \Omega_t + \tfrac{K}{r}(\psi_\theta \Omega_r - \psi_r \Omega_\theta) &= Kw(w_r \sin\theta + \tfrac{1}{r}w_\theta \cos\theta) + \nabla^2\Omega \end{aligned}$$

Assuming a core/boundary layer structure with $p \gg 1$, so that $\nabla^2 \approx \frac{\partial^2}{\partial r^2}$, the equations are rescaled according to

$$T = p^2 t, \qquad W = wp^2, \qquad \Psi = p^7\psi/K, \qquad n = (1-r)p$$

to obtain in the boundary layer near $r = 1$

$$\begin{aligned} W_T + \varepsilon(\Psi_n W_\theta - \Psi_\theta W_n) &= \cos 2T + W_{nn} \\ \Psi_{nnT} + \varepsilon(\Psi_n \Psi_{nn\theta} - \Psi_\theta \Psi_{nnn}) &= WW_n \sin\theta + \Psi_{nnnn} \end{aligned}$$

Identify the constant ε which derives from the rescaling. An appropriate set of boundary conditions are

$$W = \Psi = \Psi_n = 0 \qquad \text{on} \quad n = 0, \qquad \Psi_{nn} \to 0, \quad W_n \to 0 \qquad \text{as} \quad n \to \infty.$$

Setting $\varepsilon = 0$, seek a solution with $W = \Re e\left[f(n)e^{2iT}\right]$, where $\Re e$ denotes the real part, and deduce that

$$W = \tfrac{1}{2}\sin 2T - \tfrac{1}{2}\sin(2T - n)e^{-n}\ .$$

Defining a time average of a quantity ξ by $\overline{\xi} = \frac{1}{\pi}\int_0^\pi \xi\, dT$, show that W^2 has the average

$$\overline{W^2} = \tfrac{1}{2}|f|^2 = \tfrac{1}{8} + \tfrac{1}{8}e^{-2n} - \tfrac{1}{4}e^{-n}\cos n\ .$$

Take an average of the Ψ-equation and verify that the solution is

$$\overline{\Psi_n} = \tfrac{1}{64}\sin\theta(1 - e^{-2n} - 4e^{-n}\sin n)\ .$$

What does this imply about the flow in the core of the pipe? Discuss briefly the resultant advantages in an industrial or physiological context.

3. Give an account of any topic covered in the course not explicitly examined on this paper.

Discuss, as appropriate, the industrial significance of the problem, the assumptions of the model and the solution.

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4. A uniform magnetic field $\mathbf{B} = B_0(\cos \Omega t,\, \sin \Omega t,\, 0)$ (in Cartesian coordinates) rotates slowly about the z-axis. As it passes through a region of fluid with permeability μ_0 and conductivity σ, it generates a current density $\mathbf{j}$ and electric field $\mathbf{E}$ according to the equations

$$\mathbf{j} = \sigma \mathbf{E}, \qquad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} \ .$$

If the fluid region has typical dimension L, find a characteristic scale Ω_0 such that if $\Omega \ll \Omega_0$ then the magnetic field is spatially uniform at leading order.

In this case, show that the rate of vorticity generation within the fluid takes the constant value

$$\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) = (0,\, 0,\, \sigma \Omega B_0^2) \ .$$

A vertical column of solidifying liquid metal occupies $r < a$ in terms of cylindrical polar coordinates $(r,\, \theta,\, z)$. Assuming the flow has circular streamlines, $\mathbf{u} = (0,\, u_\theta(r),\, 0)$ (in cylindrical coordinates) the Navier-Stokes equations require

$$\mu \nabla^2 \boldsymbol{\omega} + \nabla \wedge (\mathbf{j} \wedge \mathbf{B}) = 0 \qquad \text{where} \quad \boldsymbol{\omega} = \left(0,\, 0,\, \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}\right) \ .$$

Find $u_\theta(r)$ given that the boundary $r = a$ is solid. Can the fluid rotation rate exceed Ω?

Without performing any calculation, state roughly what value of Ω gives maximum stirring when the other parameters are fixed.

$$\left[\text{In cylindrical polars} \qquad \nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \ .\right]$$

Figure 1: Geometry of the flow described in Question 5

5. Consider a multilayer material flowing downslope on a flat plate that is inclined at an angle ϕ to the horizontal. We assume that the flow is driven by gravity and that surface tension is not important. Let us choose axes such that x is downslope and z is perpendicular to the plate. The geometry is shown in the figure.

The multilayer fluid is composed of two immiscible, different Newtonian (incompressible) fluids. These fluids have viscosity μ_k and density ρ_k, where $k = 1, 2$ refers respectively to the upper and lower fluid layers. Let the thickness of the upper fluid be $\Theta(x, t)$ and that of the lower fluid be $\zeta(x, t)$; the total thickness is $h(x, t) = \Theta(x, t) + \zeta(x, t)$.

The governing equations are:

$$\rho_k(\partial_t u + u\partial_x u + w\partial_z u) = -\partial_x p + (\partial_x \tau_{xx} + \partial_z \tau_{xz}) + \rho_k g \sin\phi,$$

$$\rho_k(\partial_t w + u\partial_x w + w\partial_z w) = -\partial_z p + (\partial_x \tau_{xz} + \partial_z \tau_{zz}) - \rho_k g \cos\phi$$

$$\partial_x u + \partial_z w = 0.$$

Where $k = 1, 2$ depending whether we are in the upper or lower fluid. The ∂_x, ∂_z denote partial derivatives with respect to x and z, and τ_{ij} denotes the

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deviatoric stresses. Since the fluids are Newtonian, what is the stress-strain rate relation in each fluid?

You are required to non-dimensionalize the governing equations using L, H as typical lengthscales in the x, z directions, where $\epsilon = H/L \ll 1$. Choose the velocity scale U for u, and pressure scale P for p to be

$$U = \frac{\rho_1 g H^3 \cos\phi}{\mu_2 L}, P = \rho_1 g H \cos\phi.$$

Justify this choice of scales, that is, what effects do they balance?

Use the viscosity of fluid 2 as the reference viscosity when non-dimensionalizing the stresses, and show that the non-dimensionalized equations (the hat decoration on the variables denotes that they are non-dimensionalized) are:

$$\epsilon^2 D_k Re(\partial_{\hat{t}}\hat{u} + \hat{u}\partial_{\hat{x}}\hat{u} + \hat{w}\partial_{\hat{z}}\hat{u}) = -\partial_{\hat{x}}\hat{p} + \epsilon\partial_{\hat{x}}\hat{\tau}_{\hat{x}\hat{x}} + \partial_{\hat{z}}\hat{\tau}_{\hat{x}\hat{z}} + SD_k$$

$$\epsilon^4 D_k Re(\partial_{\hat{t}}\hat{w} + \hat{u}\partial_{\hat{x}}\hat{w} + \hat{w}\partial_{\hat{z}}\hat{w}) = -\partial_{\hat{z}}\hat{p} + \epsilon^2\partial_{\hat{x}}\hat{\tau}_{\hat{x}\hat{z}} + \epsilon\partial_{\hat{z}}\hat{\tau}_{\hat{z}\hat{z}} - D_k$$

for $k = 1, 2$, and

$$\partial_{\hat{x}}\hat{u} + \partial_{\hat{z}}\hat{w} = 0.$$

Given that $D_k = \rho_k/\rho_1$, identify the non-dimensional parameters S, Re. Physically, what would you expect to happen if $\rho_2 < \rho_1$?