UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

MSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2004

M4A33/MSA3 Applications of Fluid Mechanics

DATE: Tuesday 8^{th} June 2004 TIME: 10 a.m. – 12 noon

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. The heat equation for a temperature field $T(x,t)$ in $x > 0$ is

$$T_t = \kappa T_{xx} \; .$$

There are no phase changes. It is subject to boundary conditions

$$T_x = -f(T_s)$$

at $x = 0$ where $T_s(t) = T(0,t)$ and f is a given function of temperature, and $T \to T_i$ as $x \to \infty$ and $T = T_i$ at $t = 0$.

Consider the integral balance method, setting

$$T = B + A\left(1 - \frac{x}{\delta(t)}\right)^2$$

in a thermal boundary layer of width $\delta(t)$. Using appropriate boundary conditions on $x = \delta$ determine A, B and δ in terms of $T_i, T_s, f(T_s)$ and κ.

Show that $T_s(t)$ is the solution of the transcendental equation

$$6\kappa t = \int_{T_i}^{T_s} \frac{4(T_s - T_i)}{f^3(T_s)}[2f(T_s) - (T_s - T_i)f'(T_s)]dT_s.$$

Further, setting $f(T_s) = 1$ show that $T_s(t) = T_i + \left(\frac{3\kappa t}{2}\right)^{\frac{1}{2}}$ and that the thermal boundary layer thickness δ grows as $(Ct)^{\frac{1}{2}}$ for some constant C to be determined.

Why would you anticipate the $\sqrt{t}$ dependence in the boundary layer thickness? Consider $F = T_x$ as a new variable and show that the problem considered here, for F with $f(T_s) = 1$, is self-similar. Construct the solution for $T_x(x,t)$ and hence find that

$$T_s(t) = T_i + \frac{2\sqrt{\kappa t}}{\sqrt{\pi}}, \quad \text{you may use that} \quad \int_0^\infty \text{erfc}(\tau)d\tau = 1/\sqrt{\pi}.$$

2. Fluid flow in the channel $0 < y < a$ can be modelled by a steady Poiseuille velocity in the x-direction,

$$V(y) = \beta Y(1-Y) \qquad \text{for some constant } \beta \text{ where } Y = y/a.$$

Defining y-averages over the channel cross-section for some quantity w by

$$\overline{w}(x,\,t) = \frac{1}{a}\int_0^a w(x,\,y,\,t)\,dy = \int_0^1 w\,dY\ ,$$

show that $\beta = 6\overline{V}$. A chemical is carried passively by the fluid so that its concentration $u(x, y, t)$ satisfies

$$u_t + V(y)u_x = Du_{xx} + Du_{yy} \qquad \text{with} \quad u_y = 0 \qquad \text{on} \quad y = 0, a\ .$$

Writing

$$u(x,\,y,\,t) = \overline{u}(x,\,t) + u'(x,\,y,\,t) \qquad \text{where} \quad \overline{u'} = 0,$$

show that

$$\overline{u}_t + \overline{V}\,\overline{u}_x + \overline{Vu'_x} = D\overline{u}_{xx}\ .$$

Explaining carefully the physical arguments for the approximations, show that after a time long compared with a^2/D,

$$u' \simeq \frac{\overline{V}\,\overline{u}_x a^2}{D}\left(Y^3 - \tfrac{1}{2}Y^4 - \tfrac{1}{2}Y^2 + \tfrac{1}{60}\right) \qquad \text{where} \quad Y = y/a.$$

Deduce that the effective shear-enhanced diffusion coefficient is

$$D_{\text{eff}} = D + \frac{\overline{V}^2 a^2}{210D}\ .$$

Discuss briefly the physical processes involved and the practical significance of this result when the diffusivity D is small. [N.B. You may find the result $\frac{6}{5} - \frac{9}{6} + \frac{3}{7} - \frac{3}{4} + \frac{3}{5} + \frac{1}{20} - \frac{1}{30} = -\frac{1}{210}$ useful. More marks will be awarded for correct arguments than for correct arithmetic.]

Turn over...

3. Give an account of any topic covered in the course not explicitly examined on this paper. (For example, solidification or melting, surfactant transport, flow in curved pipes, electrically charged drops etc.)

Discuss, as appropriate, the practical significance of the problem, the assumptions of the model and the solution.

4. Liquid metal occupies the slab $0 > y > -b$. The region $0 < y < a$ contains insulating gas, and a set of electromagnets at $y = a$ generate an alternating magnetic field which travels as a wave in the x-direction.
Writing $\mathbf{B} = \nabla \wedge (0,\, 0,\, \psi(x,\, y,\, t))$, the governing equations for ψ are

$$\nabla^2 \psi = 0 \qquad \text{in} \quad 0 < y < a\ , \qquad \nabla^2 \psi = \mu_0 \sigma \frac{\partial \psi}{\partial t} \qquad \text{in} \quad 0 > y > -b$$

where σ and μ_0 are respectively the conductivity and permeability of the metal. On $y = a$, it is given that $\psi = \Re e[A e^{i(kx - \omega t)}]$, where A is a known complex constant and $\Re e$ denotes the real part. On $y = -b$ it may be assumed that $\psi \simeq 0$ and ψ and ψ_y are continuous at $y = 0$.

Defining the skin-depth

$$\delta = \left(\frac{2}{\omega \mu_0 \sigma} \right)^{1/2}$$

and assuming $k\delta \ll 1$ and $\delta \ll b$, find $\psi(x,\, y)$ in the metal. Show that the time-averaged Lorentz force, $\mathbf{F} \equiv \frac{\omega\sigma}{2} \Re e\, [i\psi \nabla \psi^*]$ where * denotes the complex conjugate, is given by

$$\mathbf{F} \simeq C e^{2y/\delta} (k,\, -1/\delta,\, 0) \qquad \text{where} \quad C = \frac{k^2 |A|^2}{2\mu_0 \sinh^2(ka)}.$$

As $\mathbf{F}$ depends only on y, it drives steady unidirectional flow $\mathbf{u} = (u(y),\, 0,\, 0)$. Assuming the metal is a Newtonian fluid of viscosity μ and that no pressure gradient acts in the x-direction, find $u(y)$ given that that $u = 0$ on $y = -b$ and $\partial u / \partial y = 0$ on $y = 0$.

Sketch the velocity profile.

5. Consider the slumping of a finite volume of Newtonian fluid, fed at a constant rate so the volume flux is constant, down a constant incline.

After non-dimensionalization we arrive at an evolution equation for the height field, $h(x, y, t)$,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\left(S - \frac{\partial h}{\partial x}\right)\right) - \frac{\partial}{\partial y}\left(\frac{h^3}{3}\frac{\partial h}{\partial y}\right) = 0.$$

The x co-ordinate points downslope and y is cross-slope. The slope parameter, S, is zero for a horizontal plane.

Given this evolution equation, use similarity variables to obtain scalings for the maximal height, $h(0, 0, t)$, the maximal downslope, $X_+(t)$, and maximal cross-slope, $Y_+(t)$, positions when $S \neq 0$ (justifying the omission of any terms) and when $S = 0$.

For a real flow starting from an axisymmetric mound of fluid which scalings are followed by $X_+(t)$ initially, and at a later stage? Sketch $X_+(t), Y_+(t), h(0, 0, t)$ versus t on log-log axes.