M4A33 Industrial Mathematics Brief Notes on Electrodynamics

Electric charge is a conserved quantity which can have either sign. We use a continuum approach, with an electric charge density $\rho(\mathbf{x},t)$, so that the total charge in a volume V is

$$Q(t) = \int_V \rho(\mathbf{x},t)dV \tag{1.2}$$

Flux of charge consistutes electric currents. A charge density ρ moving with velocity $\mathbf{v}$ constitues an electric current density, $\mathbf{j} = \rho\mathbf{v}$ (1.3). Note that because electrons can move relative to V sitively charged ions frequently a current density $\mathbf{j}$ exists even though the net charge density $\rho = 0$. The current I flowing across an area A with unit normal $\widehat{\mathbf{n}}$ is

$$I = \int_A \mathbf{j}\cdot\widehat{\mathbf{n}}\,dS \tag{1.4}$$

The conservation of charge is expressed by the **charge conservation law**

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0\ . \tag{1.5}$$

Charges and currents give rise to an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$. These quantities are related by **Maxwell's equations**, which in S.I. units are

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon \tag{1.6}$$

$$\nabla\wedge\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{1.7}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{1.8}$$

$$\nabla\wedge\mathbf{B} = \mu\left(\mathbf{j} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}\right) \tag{1.9}$$

The equations include two physical quantities which are assumed constant for the medium in question, the permittivity ε and pemeability μ. In a vacuum $\varepsilon = \varepsilon_0 \simeq 8.85\times10^{-12}$ and $\mu = \mu_0 = 4\pi\times10^{-7}$. The speed of light in a vacuum, c, is given by $c^2\varepsilon_0\mu_0 = 1$.

Notes: (1) (1.6) relates the total charge in a volume to the total electric field coming out (Gauss' Law) $Q = \varepsilon\oint\mathbf{E}\cdot\widehat{\mathbf{n}}\,dS$. (1.10)

(2) (1.7) is Faraday's Law of induction. Changing magnetic fields generate electric fields and potentials. If $\partial\mathbf{B}/\partial t = 0$, we have **electrostatics**, when we write

$$\mathbf{E} = -\nabla\phi(\mathbf{x}) \qquad \text{and} \quad \nabla^2\phi = -\rho/\varepsilon \tag{1.11}$$

(3) The lack of a RHS in (1.8) means there are no "magnetic charges" (or monopoles). Magnetic dipoles exist, however (think of a little bar magnet)

(4) The last term in (1.9) is called the **displacement current** and in many circumstances it can be neglected, as it is a relativistic correction. $\nabla\wedge\mathbf{B} = \mu\mathbf{j}$ is **Ampère's Law**. The magnetic vector potential $\mathbf{A}$ is then defined by

$$\mathbf{B} = \nabla\wedge\mathbf{A}, \qquad \nabla\cdot\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} = -\mu\mathbf{j} \tag{1.12}$$

(5) Maxwell's equations are relativistic, so that they remain invariant if we make a Lorentz transformation of $(\mathbf{x}, t)$. We shall not deal with relativity in this course, but we need to know how the fields seen by a moving charge relate to the ones seen in the laboratory frame. A particle moving with speed $\mathbf{v}$ sees fields

$$\mathbf{B}^{part} = \mathbf{B}^{lab}, \qquad \mathbf{E}^{part} = \mathbf{E}^{lab} + \mathbf{v} \wedge \mathbf{B}^{lab} \tag{1.21}$$

A moving charge q feels a force $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$ (1.22), so that the force density on a continuum is

$$\mathbf{F} = \rho\mathbf{E} + \mathbf{j} \wedge \mathbf{B} \tag{1.24}$$

The last term in (1.24) is called the **Lorentz force**.

Ohm's Law: The extent to which an electric field can cwcauseget charges to move in a medium is represented by the **conductivity** σ, so that in a stationary medium $\mathbf{j} = \sigma\mathbf{E}$ or in general

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{1.23}$$

If $\sigma = 0$, we say the medium is an insulator, otherwise a conductor. An initial charge distribution $\rho(\mathbf{x}, 0)$ in a conductor decays to zero on the charge relaxation time-scale $\tau = \varepsilon/\sigma$, as the free charges combine or move away towards the boundary of the conducting region.

At a boundary between two different media, we must allow for surface current density $\mathbf{j}_s$ and a surface charge density ρ_s. Across an interface where the electric properties $\varepsilon_1, \mu_1, \sigma_1$ change to $\varepsilon_2, \mu_2, \sigma_2$, Maxwell's equations require

$$[\mathbf{B} \cdot \widehat{\mathbf{n}}] = 0, \qquad [\widehat{\mathbf{n}} \wedge \mathbf{E}] = 0, \qquad [\mathbf{j} \cdot \widehat{\mathbf{n}}] = 0, \qquad [\varepsilon\mathbf{E} \cdot \widehat{\mathbf{n}}] = \rho_s, \qquad [\widehat{\mathbf{n}} \wedge \mathbf{B}/\mu] = \mathbf{j}_s \tag{1.20}$$

where the square brackets denote the change in values across the interface, so that for example $\rho_s = \varepsilon_2\mathbf{E}_2 \cdot \widehat{\mathbf{n}} - \varepsilon_1\mathbf{E}_1 \cdot \widehat{\mathbf{n}}$. Thus normal $\mathbf{B}$ and tangential $\mathbf{E}$ are continuous. This latter condition means the electric potential ϕ is continuous everywhere.

The stress on a surface can be found using the Maxwell stress tensor, T_{ij}, given by

$$T_{ij} = \varepsilon(E_iE_j - \tfrac{1}{2}\delta_{ij}|\mathbf{E}|^2) + (1/\mu)(B_iB_j - \tfrac{1}{2}\delta_{ij}|\mathbf{B}|^2) \tag{1.25}$$

The stress (or traction) on a surface with normal $\widehat{\mathbf{n}}$ is $[T_{ij}n_j]$. Jumps in electric properties tend to lead to forces on the interface. Exercise: If the fields are steady, show that

$$F_i = \frac{\partial T_{ij}}{\partial x_j}\ . \tag{1.26}$$

The rate of working by the force $\mathbf{f}$ is $w = \mathbf{f} \cdot \mathbf{v}$. This work is converted into heat, so that the Ohmic decay, or Joule heating rate density W is given by

$$W = \mathbf{F} \cdot \mathbf{v} = \mathbf{j} \cdot \mathbf{E} \tag{1.27}$$

If $\mathbf{v} = 0$, then $W = |\mathbf{j}|^2/\sigma$.