M4A33; TIG-welding: Current source in liquid metal half-space

In the TIG (Tungsten/Inert Gas) welding process, two metallic surfaces are brought into contact, and a strong electric current is introduced at the join. The resultant heating melts the metal in the vicinity of the electrodea .The electrode passes slowly along the interface and the liquid metal solidifies behind it, thus welding the metals together.

We model the behaviour in the weldpool by considering an infinite half-space of liquid metal, occupying $0 < \theta < \frac{\pi}{2}$ in terms of spherical polar coordinates (r, θ, η), with an axis pointing into the metal. (Note: in lectures the axis pointed up, so that $\theta \to \pi - \theta$.) A constant electric current I enters the liquid metal pool at the origin. The metal has kinematic viscosity ν and density ρ.

As the configuration is axisymmetric, we expect the magnetic field $\mathbf{B}$, current density $\mathbf{j}$ and velocity $\mathbf{u}$ also to be independent of η. Furthermore, $\mathbf{j}$ will be entirely radial and a function only of r. [As the magnetic Reynolds number is small, we can neglect the magnetic induction term $\mathbf{u} \wedge \mathbf{B}$. Then as $\mathbf{j} = \sigma \mathbf{E}$ and $\nabla \wedge \mathbf{E} = 0$, we have $\nabla \wedge \mathbf{j} = 0$ also] $\mathbf{B}$ will therefore be in the η-direction (*azimuthal* or *toroidal*), and the Lorentz force $\mathbf{j} \wedge \mathbf{B}$ acts in the (r, θ) (or *poloidal*) plane. As this force is **rotational** ($\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) \neq 0$), it cannot be balanced by a pressure gradient and so necessarily drives fluid motion. As the driving force is poloidal, we might expect the velocity also to be poloidal. In that case it can be represented by a streamfunction $\psi(r, \theta)$. We therefore write

$$\mathbf{j} = (j(r), 0, 0) \qquad \mathbf{B} = (0, 0, B(r, \theta)) \qquad \mathbf{u} = \nabla \wedge \left(0, 0, \frac{\psi(r, \theta)}{r \sin\theta}\right) . \tag{1}$$

We will make frequent use below of the following formulae: In terms of spherical polar coordinates (r, θ, η), an axisymmetric vector $\mathbf{F}(r, \theta) = (F_r, F_\theta, F_\eta)$ satisfies

$$\nabla \cdot (F_r, F_\theta, F_\eta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_\theta\right) . \tag{2}$$

$$\nabla \wedge (F_r, F_\theta, F_\eta) = \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\eta \sin\theta),\ -\frac{1}{r}\frac{\partial}{\partial r}(F_\eta r),\ \frac{1}{r}\frac{\partial(rF_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\right] \tag{3}$$

Firstly, we note that by charge conservation $\nabla \cdot \mathbf{j} = 0$, so that for some constant α

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j) = 0 \qquad \Longrightarrow \qquad j = \frac{\alpha}{r^2} = \frac{I}{2\pi r^2} , \tag{4}$$

since the total current flowing across a hemisphere around the origin is I. Using Ampère's Law, $\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$,

$$\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(B\sin\theta),\ -\frac{1}{r}\frac{\partial}{\partial r}(rB),\ 0\right) = \left(\frac{\mu_0 I}{2\pi r^2},\ 0,\ 0\right)$$

so that $B = b(\theta)/r$ where, as b must be finite on the axis $\theta = 0$,

$$\frac{1}{r^2 \sin\theta}\frac{\partial(b\sin\theta)}{\partial\theta} = \frac{\mu_0 I}{2\pi r^2} \qquad \Longrightarrow \qquad b = \frac{\mu_0 I(1-\cos\theta)}{2\pi\sin\theta}. \tag{5}$$

Combining (4) and (5), $\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) = (0,\, 0,\, G)$ where

$$G = \frac{\mu_0 I^2}{2\pi^2}\left(\frac{1-\cos\theta}{r^4 \sin\theta}\right) = \frac{\rho\nu^2 K \sin\theta}{r^4(1+\cos\theta)} \qquad \text{where} \quad K = \frac{\mu_0 I^2}{2\pi^2\nu^2\rho} \ . \tag{6}$$

Since the driving term $\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) \sim r^{-4}$, and there is no natural length-scale in the problem, we seek a similarity solution with $\mathbf{u} \sim r^{-1}$ and $\underline{\omega} \equiv \nabla \wedge \mathbf{u} \sim r^{-2}$. With this scaling both $\nabla \wedge (\mathbf{u} \wedge \underline{\omega})$ and $\nabla^2 \underline{\omega}$ will scale as r^{-4} also. Writing $\psi = \nu r f(\theta) \equiv \nu r g(\mu)$, where $\mu = \cos\theta$, so that $\frac{\partial}{\partial\theta} \equiv -\sin\theta \frac{\partial}{\partial\mu}$, we have, from (1),

$$\mathbf{u} = \frac{\nu}{r\sin\theta}\left(\frac{\partial f}{\partial\theta}, -f, 0\right) = \frac{\nu}{r}\left(-g', -\frac{g}{\sin\theta}, 0\right) \qquad \text{and} \quad \underline{\omega} = -\frac{\nu}{r^2}\left(0,\, 0,\, \sin\theta\, g''\right) \ , \tag{7}$$

where $'$ denotes differentiation with respect to μ. Thus $\mathbf{u} \wedge \underline{\omega} = \frac{\nu^2}{r^3}(gg'', -\sin\theta g'g'', 0)$ and

$$\nabla \wedge (\mathbf{u} \wedge \underline{\omega}) = \frac{\nu^2 \sin\theta}{r^4}\left(0,\, 0,\, 2g'g'' + (gg'')'\right) \ . \tag{8}$$

Similarly, using (3) twice, $\nabla \wedge \underline{\omega} = \frac{\nu}{r^3}\left([\sin^2\theta\, g'']', -\sin\theta\, g'', 0\right)$ and as $\sin^2\theta = 1 - \mu^2$,

$$\nabla^2 \underline{\omega} = -\nabla \wedge (\nabla \wedge \underline{\omega}) = \frac{\nu \sin\theta}{r^4}\Big(0,\, 0,\, \big[4\mu g''' - (1-\mu^2)g''''\big]\Big) \ . \tag{9}$$

Thus, from (6), (8), and (9), the vorticity equation, $\nabla \wedge (\mathbf{u} \wedge \underline{\omega}) + \frac{1}{\rho}\nabla \wedge (\mathbf{j} \wedge \mathbf{B}) + \nu \nabla^2 \underline{\omega} = 0$ only has an azimuthal component, and reduces to an ODE. This equation can in fact be integrated three times so that, for arbitrary constants A , C and D

$$\left.\begin{aligned}
\frac{\nu^2 \sin\theta}{r^4}\left[2g'g'' + (gg'')' + \frac{K}{1+\mu} - (1-\mu^2)g'''' + 4\mu g'''\right] &= 0 \\
(g')^2 + gg'' + K\ln(1+\mu) - (1-\mu^2)g''' + 2\mu g'' - 2g' &= a \\
gg' + K(1+\mu)\ln(1+\mu) - K\mu - (1-\mu^2)g'' - 2g &= a\mu + c \\
\tfrac{1}{2}g^2 - (1-\mu^2)g' - 2\mu g + \tfrac{1}{2}K(1+\mu)^2\ln(1+\mu) &= A\mu^2 + C\mu + D
\end{aligned}\right\} \tag{10}$$

Two boundary conditions are that $g(0) = 0$ and $g(1) = 0$, so that the axis and metal surface are streamlines. In addition, we need either a stress-free or no-slip condition on the surface $g = 0$. (10) can then be integrated numerically. The third boundary condition seems not to affect the flow qualitatively. It is found that the flow has a jet-like structure along the axis ($\theta = 0$ or $\mu = 1$) away from the electrode. Interestingly, for large enough K ($K > 200.1$ for the no-slip case, $g'(0) = 0$) it is found that the velocity on the axis becomes infinite ($g'(1) = \infty$). Clearly, a real fluid would avoid this singularity somehow. Experimental observations show that an axial jet is indeed formed which increases in strength as K increases. However, before the theoretical singularity is reached, the fluid starts to rotate. If we return to (1), we can allow for this in our similarity solution, including an azimuthal swirl velocity $v(\theta)/r$. We then obtain two coupled ODEs, which are found to avoid the above singularity as $K \to \infty$.