Conservative Forces and Path Independence

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

Example 1: Find $\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$, where $\underline{\mathbf{F}} = y^2\,\underline{\mathbf{i}} + x^2\,\underline{\mathbf{j}}$ and C is (a) the straight line $y = x$ and (b) the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

$$\text{(a)} \quad \Longrightarrow \quad \int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \;=\; \int_C (y^2 dx + x^2 dy) \;=\; \int_C \left(y^2 + x^2 \frac{dy}{dx} \right) dx \;=\; \int_0^1 (x^2 + x^2)\, dx \;=\; \tfrac{2}{3}. \tag{1a}$$

$$\text{(b)} \quad \Longrightarrow \quad \int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \;=\; \int_C \left(y^2 + x^2 \frac{dy}{dx} \right) dx \;=\; \int_0^1 (x^4 + x^2\, 2x)\, dx \;=\; \tfrac{7}{10}. \tag{1b}$$

$\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ is clearly **PATH DEPENDENT** in general.

Example 2: Find $\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$, where $\underline{\mathbf{F}} = (x - xy^2)\,\underline{\mathbf{i}} + (8y - x^2 y)\,\underline{\mathbf{j}}$ and C is (a) the straight line $y = x$ from $(0,0)$ to $(1,1)$ and (b) is the path firstly along $y = 0$ from $(0,0)$ to $(1,0)$, then along $x = 1$ from $(1,0)$ to $(1,1)$.

$$\text{(a)} \quad \Longrightarrow \quad \int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \;=\; \int_C \left((x - xy^2) + (8y - x^2 y)\frac{dy}{dx} \right) dx \;=\; \int_0^1 (9x - 2x^3)\, dx \;=\; 4. \tag{2a}$$

$$\text{(b)} \quad \Longrightarrow \quad \int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} \;=\; \int_C \big((x - xy^2)\, dx + (8y - x^2 y)\, dy \big) \;=\; \int_0^1 x\, dx + \int_0^1 7y\, dy \;=\; 4. \tag{2b}$$

In this case we obtain the **SAME** value for $\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ along **DIFFERENT** paths.

Given a vector field $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv F_1(\underline{\mathbf{r}})\,\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\,\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\,\underline{\mathbf{k}}$, suppose there exists a function $\phi(\underline{\mathbf{r}}) \equiv \phi(x, y, z)$ such that

$$\frac{\partial \phi}{\partial x} = F_1\,, \qquad \frac{\partial \phi}{\partial y} = F_2\,, \qquad \frac{\partial \phi}{\partial z} = F_3\,; \tag{3a}$$

then

$$\int_{C_{\mathbf{A}\to\mathbf{B}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_{\mathbf{A}\to\mathbf{B}}} \left[\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right] \;=\; \int_{C_{\mathbf{A}\to\mathbf{B}}} d\phi \;=\; \phi(\mathbf{B}) - \phi(\mathbf{A}); \tag{3b}$$

that is, $\int_{C_{\mathbf{A}\to\mathbf{B}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ depends only on the start point **A** and the end point **B**, and **NOT** on the particular path C joining **A** to **B**.

If ϕ exists satisfying (3a), then

$$\begin{aligned}
\frac{\partial F_1}{\partial y} \;&=\; \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} \right) \;=\; \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} \right) \;=\; \frac{\partial F_2}{\partial x}\,, \\
\frac{\partial F_1}{\partial z} \;&=\; \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x} \right) \;=\; \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z} \right) \;=\; \frac{\partial F_3}{\partial x}\,, \\
\frac{\partial F_2}{\partial z} \;&=\; \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y} \right) \;=\; \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z} \right) \;=\; \frac{\partial F_3}{\partial y}\,.
\end{aligned} \tag{4}$$

One can show that if $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv F_1(\underline{\mathbf{r}})\,\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\,\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\,\underline{\mathbf{k}}$ satisfies the 3 conditions in (4), i.e.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}; \tag{5}$$

then there exists ϕ satisfying (3a) and hence (3b) holds.

Such an $\underline{\mathbf{F}}$ is called **CONSERVATIVE** with the corresponding ϕ being the **POTENTIAL** of $\underline{\mathbf{F}}$.

If C is a closed path and $\underline{\mathbf{F}}$ is conservative, then $\oint_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = 0$ as $\mathbf{B} = \mathbf{A}$ in (3b).

Example 1 : $\Longrightarrow \dfrac{\partial F_1}{\partial y} = 2y \neq 2x = \dfrac{\partial F_2}{\partial x} \Longrightarrow \underline{\mathbf{F}}$ is NOT conservative

$\Longrightarrow \int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ is NOT path independent.

Example 2 : $\Longrightarrow \dfrac{\partial F_1}{\partial y} = -2xy = \dfrac{\partial F_2}{\partial x}, \quad \dfrac{\partial F_1}{\partial z} = 0 = \dfrac{\partial F_3}{\partial x}, \quad \dfrac{\partial F_2}{\partial z} = 0 = \dfrac{\partial F_3}{\partial y}$

$\Longrightarrow \underline{\mathbf{F}}$ is conservative $\Longrightarrow \int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ is path independent.

How to find the corresponding potential ϕ satisfying (3a), i.e.

$$\text{(a)}\ \frac{\partial \phi}{\partial x} = x - xy^2, \quad \text{(b)}\ \frac{\partial \phi}{\partial y} = 8y - x^2 y, \quad \text{(c)}\ \frac{\partial \phi}{\partial z} = 0.$$

(a) $\Longrightarrow \phi(x,y,z) = \frac{1}{2}x^2 - \frac{1}{2}x^2y^2 + f(y,z)$ (f an arbitrary function of y and z)

$\Longrightarrow \dfrac{\partial \phi}{\partial y} = -x^2 y + \dfrac{\partial f}{\partial y}(y,z) = 8y - x^2 y$ from (b)

$\Longrightarrow \dfrac{\partial f}{\partial y}(y,z) = 8y \Longrightarrow f(y,z) = 4y^2 + g(z)$ (g an arbitrary function of z)

$\Longrightarrow \phi(x,y,z) = \frac{1}{2}x^2(1-y^2) + 4y^2 + g(z) \Longrightarrow \dfrac{\partial \phi}{\partial z} = \dfrac{dg}{dz}(z) = 0$ from (c)

$\Longrightarrow g(z) = \phi_0$ (ϕ_0 an arbitrary constant)

$\Longrightarrow \phi(x,y,z) = \frac{1}{2}x^2(1-y^2) + 4y^2 + \phi_0$.

Hence in **Example 2:** $\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \phi(1,1,0) - \phi(0,0,0) = 4$ as in (2a) and (2b).

Example 3: Show that $\underline{\mathbf{F}} = 2xy\,\underline{\mathbf{i}} + (x^2 + 2y\sin z)\,\underline{\mathbf{j}} + y^2\cos z\,\underline{\mathbf{k}}$ is conservative and find the corresponding potential ϕ.

$\dfrac{\partial F_1}{\partial y} = 2x = \dfrac{\partial F_2}{\partial x}, \quad \dfrac{\partial F_1}{\partial z} = 0 = \dfrac{\partial F_3}{\partial x}, \quad \dfrac{\partial F_2}{\partial z} = 2y\cos z = \dfrac{\partial F_3}{\partial y} \Longrightarrow \underline{\mathbf{F}}$ is conservative.

$\dfrac{\partial \phi}{\partial x} = F_1 = 2xy \Longrightarrow \phi(x,y,z) = x^2 y + f(y,z)$ (f an arbitrary function of y and z)

$\Longrightarrow \dfrac{\partial \phi}{\partial y} = x^2 + \dfrac{\partial f}{\partial y}(y,z) = F_2 = x^2 + 2y\sin z \Longrightarrow \dfrac{\partial f}{\partial y}(y,z) = 2y\sin z$

$\Longrightarrow f(y,z) = y^2\sin z + g(z)$ (g an arbitrary function of z)

$\Longrightarrow \phi(x,y,z) = x^2 y + y^2\sin z + g(z) \Longrightarrow \dfrac{\partial \phi}{\partial z} = y^2\cos z + \dfrac{dg}{dz}(z) = F_3 = y^2\cos z$

$\Longrightarrow g(z) = \phi_0$ ϕ_0 an arbitrary constant

$\Longrightarrow \phi(x,y,z) = x^2 y + y^2\sin z + \phi_0$.