M.Eng. 2.6 Mathematics: Grad, Div and Curl

This sheet can be found on the Web: http://www.ma.ic.ac.uk/∼ajm8/MEng26

A function $\underline{\mathbf{F}}$ which associates a vector value $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv \underline{\mathbf{F}}(x, y, z) \equiv F_1(\underline{\mathbf{r}})\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\underline{\mathbf{k}}$ with every position vector $\underline{\mathbf{r}} \equiv x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$ is said to be a **VECTOR FIELD**, e.g. velocity of a fluid.

Recall if $f(\underline{\mathbf{r}})$ is a scalar field, then

$$\text{the gradient of } f \equiv \text{grad}\, f \equiv \underline{\nabla} f \equiv \frac{\partial f}{\partial x}\underline{\mathbf{i}} + \frac{\partial f}{\partial y}\underline{\mathbf{j}} + \frac{\partial f}{\partial z}\underline{\mathbf{k}}\,. \tag{1}$$

$\underline{\nabla} \equiv \underline{\mathbf{i}}\dfrac{\partial}{\partial x} + \underline{\mathbf{j}}\dfrac{\partial}{\partial y} + \underline{\mathbf{k}}\dfrac{\partial}{\partial z}$ is a **VECTOR OPERATOR**, $\quad \underline{\nabla}$: scalar field $\to$ vector field.

Now introduce

$$\text{the divergence of } \underline{\mathbf{F}} \equiv \text{div}\, \underline{\mathbf{F}} \equiv \underline{\nabla} \cdot \underline{\mathbf{F}} \equiv \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\,. \tag{2}$$

Note that

$$\underline{\nabla} \cdot \underline{\mathbf{F}} \not\equiv \underline{\mathbf{F}} \cdot \underline{\nabla} \equiv F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z}\,,$$

which is a scalar operator; i.e.

$$\underline{\mathbf{F}} \cdot (\underline{\nabla} f) \equiv F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z} \equiv (\underline{\mathbf{F}} \,.\, \underline{\nabla}) f\,.$$

Similarly, we introduce

$$\begin{aligned} \text{curl}\, \underline{\mathbf{F}} \;&\equiv\; \underline{\nabla} \times \underline{\mathbf{F}} \;\equiv\; \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &\equiv\; \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\underline{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{\mathbf{k}}\,. \end{aligned} \tag{3}$$

The equivalent notation $\text{curl}\, \underline{\mathbf{F}} \equiv \underline{\nabla} \wedge \underline{\mathbf{F}}$ is sometimes used.
Note that $\underline{\nabla} \times \underline{\mathbf{F}} \not\equiv \underline{\mathbf{F}} \times \underline{\nabla}$, which is a vector operator; i.e.

$$\begin{aligned} \underline{\mathbf{F}} \times (\underline{\nabla} f) \;&\equiv\; \left(F_2\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial y}\right)\underline{\mathbf{i}} - \left(F_1\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial x}\right)\underline{\mathbf{j}} + \left(F_1\frac{\partial f}{\partial y} - F_2\frac{\partial f}{\partial x}\right)\underline{\mathbf{k}} \\ &\equiv\; (\underline{\mathbf{F}} \times \underline{\nabla}) f\,. \end{aligned}$$

Therefore we have that

$$\begin{array}{rcl} \text{grad} : \text{scalar field} & \to & \text{vector field} \\ \text{div} : \text{vector field} & \to & \text{scalar field} \\ \text{curl} : \text{vector field} & \to & \text{vector field} \end{array}$$

RULES: For arbitrary scalar fields $f(\underline{\mathbf{r}})$, $g(\underline{\mathbf{r}})$ and vector fields $\underline{\mathbf{F}}(\underline{\mathbf{r}})$, $\underline{G}(\underline{\mathbf{r}})$; then we have that

(1) $\Longrightarrow \underline{\nabla}(f+g) \equiv (\underline{\nabla} f) + (\underline{\nabla} g)$.

(2) $\Longrightarrow \underline{\nabla} \cdot (\underline{\mathbf{F}} + \underline{G}) \equiv (\underline{\nabla} \cdot \underline{\mathbf{F}}) + (\underline{\nabla} \cdot \underline{G})$.

(3) $\Longrightarrow \underline{\nabla} \times (\underline{\mathbf{F}} + \underline{G}) \equiv (\underline{\nabla} \times \underline{\mathbf{F}}) + (\underline{\nabla} \times \underline{G})$.

(2) and (1) $\Longrightarrow$

$$\begin{aligned} \underline{\nabla} \cdot (f\,\underline{\mathbf{F}}) &\equiv \underline{\nabla} \cdot \left[(f\,F_1)\underline{\mathbf{i}} + (f\,F_2)\underline{\mathbf{j}} + (f\,F_3)\underline{\mathbf{k}} \right] \equiv \frac{\partial (f\,F_1)}{\partial x} + \frac{\partial (f\,F_2)}{\partial y} + \frac{\partial (f\,F_3)}{\partial z} \\ &\equiv \left[F_1 \frac{\partial f}{\partial x} + f \frac{\partial F_1}{\partial x} \right] + \left[F_2 \frac{\partial f}{\partial y} + f \frac{\partial F_2}{\partial y} \right] + \left[F_3 \frac{\partial f}{\partial z} + f \frac{\partial F_3}{\partial z} \right] \\ &\equiv (\underline{\nabla} f) \cdot \underline{\mathbf{F}} + f\,(\underline{\nabla} \cdot \underline{\mathbf{F}}). \end{aligned}$$

Similarly (3) and (1) $\Longrightarrow$

$$\underline{\nabla} \times (f\,\underline{\mathbf{F}}) \equiv (\underline{\nabla} f) \times \underline{\mathbf{F}} + f\,(\underline{\nabla} \times \underline{\mathbf{F}}).$$

We introduce the **LAPLACIAN** operator ∇^2 : scalar field $\to$ scalar field, defined by

$$\nabla^2 f \equiv \underline{\nabla} \cdot (\underline{\nabla} f) \equiv \underline{\nabla} \cdot \left(\frac{\partial f}{\partial x}\underline{\mathbf{i}} + \frac{\partial f}{\partial y}\underline{\mathbf{j}} + \frac{\partial f}{\partial z}\underline{\mathbf{k}} \right) \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

It is simple matter to show using the definitions (1), (2) and (3) that

$$\underline{\nabla} \times (\underline{\nabla} f) = \underline{0} \qquad \text{and} \qquad \underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) = 0.$$

RECALL: Conservative Vector Fields and Path Independence

Given a vector field $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv F_1(\underline{\mathbf{r}})\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\underline{\mathbf{k}}$, if there exists a $\phi(\underline{\mathbf{r}})$ satisfying

$$\frac{\partial \phi}{\partial x} = F_1, \qquad \frac{\partial \phi}{\partial y} = F_2, \qquad \frac{\partial \phi}{\partial z} = F_3; \tag{4a}$$

then

$$\int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} \left[\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right] = \int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} d\phi = \phi(\underline{\mathbf{B}}) - \phi(\underline{\mathbf{A}}); \tag{4b}$$

that is, $\int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ depends only on the start point $\underline{\mathbf{A}}$ and the end point $\underline{\mathbf{B}}$, NOT on the particular path C joining $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$. We showed that a ϕ satisfying (4a) exists, and hence that (4b) holds, if and only if $\underline{\mathbf{F}}(\underline{\mathbf{r}})$ satisfies the 3 conditions

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}. \tag{5}$$

Such an $\underline{\mathbf{F}}$ is called **CONSERVATIVE** with the corresponding ϕ being the **POTENTIAL** of $\underline{\mathbf{F}}$. In terms of grad and curl, (5) $\equiv \underline{\nabla} \times \underline{\mathbf{F}} = \underline{0}$ and (4a) $\equiv \underline{\mathbf{F}} = \underline{\nabla}\,\phi$. Therefore we have that

$$\boxed{\underline{\mathbf{F}} \text{ conservative} \iff \underline{\nabla} \times \underline{\mathbf{F}} = \underline{0} \iff \underline{\mathbf{F}} = \underline{\nabla}\,\phi \text{ for some } \phi \iff (4b).}$$