M.Eng. 2.6 Mathematics: Numerical solution of ODEs

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

Most differential equations are **non-linear**, and can rarely be solved exactly. However, it is usually possible to obtain arbitrarily accurate approximate solutions on computers. We will now discuss various methods for doing this.

As we are dealing with **ODEs**, there is only one independent variable, which we will now call x rather than t. All systems of ODEs can then be expressed in terms of a vector of unknowns, $\mathbf{y}(x)$ which obeys the equation

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x,\, \mathbf{y}(x)) \qquad \text{where } \mathbf{f} \text{ is a vector of known functions.}$$

We shall concentrate on the one-dimensional problem, as most of the methods we consider can easily be generalised to more dimensions. Suppose therefore that $y(x)$ solves the problem

$$\frac{dy}{dx} = f(x,\, y(x)) \qquad \text{with} \quad y(a) = b\ . \tag{1}$$

Picard Iteration: This method is more of theoretical than practical interest. However, it is on the syllabus. We note that, formally, we can integrate (1) to give

$$y(x) = b + \int_a^x f(t,\, y(t))\, dt\ . \tag{2}$$

As $y(t)$ is not known, we cannot evaluate the above integral. However, we can use it to define a sequence of functions, $y_n(x)$, as follows:

$$\begin{aligned}
y_0(x) &= b \\
y_1(x) &= b + \int_a^x f(t,\, y_0(t))\, dt \\
y_2(x) &= b + \int_a^x f(t,\, y_1(t))\, dt \\
y_{n+1}(x) &= b + \int_a^x f(t,\, y_n(t))\, dt \qquad \text{in general.}
\end{aligned}$$

Each of the functions y_n is defined in terms of the previous one, y_{n-1}, although we may or may not be able to evaluate the integrals. Suppose now that as $n \to \infty$, $y_n(x) \to y(x)$, where $y(x)$ is some function. In that case, $y(x)$ is the solution to (2).

Example: Consider the problem

$$y' = 2xy \qquad \text{with} \quad y(0) = 1\ . \tag{3}$$

[Exercise: Show that the exact solution to this problem is $y(x) = e^{x^2}$.]
Then in integral form,

$$y(x) = 1 + \int_0^x 2t\, y(t)\, dt\ .$$

Proceeding as above, $y_0(x) = 1$,

$$y_1(x) = 1 + \int_0^x 2t(1)dt = 1 + x^2$$
$$y_2(x) = 1 + \int_0^x 2t(1+t^2)dt = 1 + x^2 + \tfrac{1}{2}x^4 \qquad \text{and eventually}$$
$$y_n(x) = 1 + x^2 + \tfrac{1}{2}x^4 + \tfrac{1}{6}x^4 + \ldots + \frac{1}{n!}x^{2n} \ .$$

Letting $n \to \infty$, we have the solution

$$y(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = e^{x^2} \ .$$

Finite Difference Methods

It is important to realise that most ODEs do not have a solution in terms of simple functions. However, they will have solutions which can be expressed as graphs, say. How can we obtain an approximation to the curves on these curves? A simple idea is to try to find some points which lie on, or close to, these graphs and join them up. So choose a **step-length** h, and define the values $x_n = a + nh$, where $n = 0, 1, 2 \ldots$ The exact solution at the value $x = x_n$ can be written $y(x_n)$. We shall now seek some values y_n which we hope will approximate the real values. To do this we must translate the differential equation into an approximate equation valid at the discrete points $\{x_n\}$.

We recall the Taylor series for the function $y(x+h)$:

$$y(x+h) = y(x) + hy'(x) + \tfrac{1}{2}h^2y''(x) + \tfrac{1}{6}h^3y'''(x) + \tfrac{1}{24}h^4y''''(x) + \ldots$$

Assuming h is small, we can therefore approximate the derivative $y'(x)$ by

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \tfrac{1}{2}hy''(x) + O(h^2) \ .$$

Here $O(h^2)$ denotes things which tend to zero as $h \to 0$ at least as fast as h^2. Now if we take $x = x_n$, then by definition $x + h = x_n + h = x_{n+1}$ and so

$$y'(x_n) = \frac{y(x_{n+1}) - y(x_n)}{h} + O(h) \simeq \frac{y_{n+1} - y_n}{h}$$

assuming that y_n is a good approximation to $y(x_n)$. We can use this last approximation in (1), to define the values y_n as the solution to

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n) \qquad \text{or} \quad y_{n+1} = y_n + hf(x_n, y_n) \qquad \text{with} \quad y_0 = b \ . \tag{4}$$

This last relation is known as **Euler's method**.