M.Eng. 2.6 Mathematics: Fourier Series.

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

A function $f(x)$ is called "$2L$-periodic" if $f(x) = f(x + 2L)$ for all x. Such a function can be represented as a **Fourier series:**

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \ , \tag{1}$$

where the constants a_n and b_n are called the **Fourier coefficients** of $f(x)$. These can be calculated from the formulae

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \ . \tag{2}$$

The factor of $\frac{1}{2}$ in (1) is included so that the formula (2) holds for $n = 0$ also. It is easiest to deal with 2π-periodic functions ($L = \pi$). The formulae for the coefficients can be derived using the **orthogonality relations:**

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

where m and n are integers with $m \neq n$. If $m = n \neq 0$, then

$$\int_{-\pi}^{\pi} \sin nx \cos nx \, dx = 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} \sin^2 nx \, dx = \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$$

Example: Consider the "square-wave" function

$$h(x) = \begin{cases} +1 & \text{for} \quad 0 < x < \pi \\ -1 & \text{for} \quad -\pi < x < 0 \end{cases}$$

Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \cos nx \, dx + \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nx \, dx = 0 \\ b_n &= \frac{1}{\pi}\int_0^{\pi} \sin nx \, dx + \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nx \, dx \\ &= \frac{1}{\pi n}\left(\Big[-\cos nx \Big]_0^{\pi} - \Big[-\cos nx \Big]_{-\pi}^{0} \right) = \frac{2}{\pi n}(1 - \cos n\pi) \ . \end{aligned}$$

Now $\cos n\pi = (-1)^n$, and so $b_n = 0$ if n is even, and $b_n = 4/(\pi n)$ if n is odd. The Fourier series for $h(x)$ is thus

$$h(x) = \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin nx}{n} = \begin{cases} +1 & \text{for} \quad 0 < x < \pi \\ -1 & \text{for} \quad -\pi < x < 0 \end{cases} \tag{3}$$

We saw from the computer demonstration that this series really does converge to the "square wave" function, although it had some difficulty at the discontinuity at $x = 0$. We can see from the series that at $x = 0$ the series sums to zero as all the terms are zero. This is a particular case of the result:

Behaviour at discontinuities: If the function $f(x)$ is discontinuous at the point $x = x_0$, taking the value f^+ as x approaches x_0 from above, and the value f^- as x approaches x_0 from below, then the series converges to the average, $\frac{1}{2}(f^+ + f^-)$.

Odd and even functions: Why were the cosine coefficients $a_n = 0$ in the above example? This was because $h(x)$ was an *odd* function: $h(-x) = -h(x)$. Thus as cos is an *even* function, $h(x)\cos(n\pi x/L)$ is an odd function. If we integrate an odd function between $-L$ and $+L$ the areas under the curve obviously cancel, and we are left with zero. Similarly, suppose $f(x)$ were an even function. Then $f(x)\sin(n\pi x/L)$ would be an odd function and thus $b_n = 0$ in that case. **Even functions only have cosines and odd functions only have sines** in their Fourier series.

Half-Range Sine & Cosine Series: Suppose $f(x)$ is defined only in $0 < x < L$. Then if we assume $f(x)$ is even, we can extend the definition to $-L < x < L$ and find its Fourier series which will have cosines only. Likewise, if we assume $f(x)$ is odd, then we can find a Fourier series with sines only. These are called **half-range series**.

Differentiation and Integration of Fourier series: If we differentiate (1) with respect to x, we find that

$$f'(x) = \sum_{n=1}^{\infty} \left[\left(\frac{-n\pi}{L} \right) a_n \sin\left(\frac{n\pi x}{L} \right) + \left(\frac{n\pi}{L} \right) b_n \cos\left(\frac{n\pi x}{L} \right) \right] , \tag{4}$$

which gives us another Fourier series for $f'(x)$. Note that differentiating brings down a factor of n, so that the coefficients of the new Fourier series are larger for large values of n, and the new series may not converge. If it does converge, however, it converges to the right answer. Likewise we can integrate (1). In that case the new series always converges.

Parseval's Theorem: What happens if we take the Fourier series for $f(x)$ and square both sides and then integrate over a period? We can use the orthogonality relations to evaluate the integrals of the product of the two series, and all these integrals are zero except when multiplying like terms together. This gives the result

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 \, dx = \tfrac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) , \tag{5}$$

which is known as Parseval's theorem, which is in some sense a generalisation of Pythagoras' theorem! The way to think about it physically is as follows: A Fourier series decomposes a signal into a sum of independent fundamental signals each with an "energy" given by the coefficient squared. Equation (5) then states that the total energy of the original signal is equal to the sum of the energies in the component parts.

If we apply (5) to the example (1), we obtain the strange result

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (1) \, dx = \sum_{\substack{n=1 \\ n\,odd}}^{\infty} \left(\frac{4}{\pi n} \right)^2 \qquad \text{or} \qquad \frac{\pi^2}{8} = 1^2 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots$$