M.Eng. 2.6: Green's Theorem in the Plane

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

GREEN'S THEOREM IN THE PLANE:

Let R be a bounded region in the $(x,\,y)$ plane with boundary C.
Let C be traversed in a **POSITIVE** direction (anticlockwise: R on your left as you follow C).
Then for all (sufficiently smooth) $f(x,y)$ and $g(x,y)$

$$\iint_R \frac{\partial f}{\partial y}(x,y)\,dx\,dy = -\oint_C f(x,y)\,dx\,, \tag{1}$$

$$\iint_R \frac{\partial g}{\partial x}(x,y)\,dx\,dy = \oint_C g(x,y)\,dy\,. \tag{2}$$

$(2)-(1) \Longrightarrow$

$$\boxed{\iint_R \left(\frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\right)dx\,dy = \oint_C \left(\,f(x,y)\,dx + g(x,y)\,dy\,\right)\,.} \tag{3}$$

Proof:

We first prove (1) for R of the special form

$$a \le x \le b, \qquad y_{\min}(x) \le y \le y_{\max}(x). \tag{4}$$

Then

$$\begin{aligned}\iint_R \frac{\partial f}{\partial y}(x,y)\,dx\,dy &= \int_a^b \left(\int_{y_{\min}(x)}^{y_{\max}(x)} \frac{\partial f}{\partial y}(x,y)\,dy\right)dx \;=\; \int_a^b \Big[\,f(x,y)\,\Big]_{y_{\min}(x)}^{y_{\max}(x)}\,dx\\ &= \int_a^b \left[\,f\big(x,y_{\max}(x)\big) - f\big(x,y_{\min}(x)\big)\,\right]dx \;=\; -\oint_C f(x,y)\,dx\,;\end{aligned}$$

that is, the desired result (1) for R of the form (4).
We now prove (2) for R of the special form

$$c \le y \le d, \qquad x_{\min}(y) \le x \le x_{\max}(y). \tag{5}$$

Then

$$\begin{aligned}\iint_R \frac{\partial g}{\partial x}(x,y)\,dx\,dy &= \int_c^d \left(\int_{x_{\min}(y)}^{x_{\max}(y)} \frac{\partial g}{\partial x}(x,y)\,dx\right)dy \;=\; \int_c^d \Big[\,g(x,y)\,\Big]_{x_{\min}(y)}^{x_{\max}(y)}\,dy\\ &= \int_c^d \left[g(x_{\max}(y),y) - g(x_{\min}(y),y)\right]dy \;=\; \oint_C g(x,y)\,dy\,;\end{aligned}$$

that is, the desired result (2) for R of the form (5).

Finally, we note that a general region R can be decomposed into regions satisfying (4) and (5). and that the integrals over the overlapping boundaries cancel. We deduce that (1), (2) and (3) hold for a general bounded region R.

Example: $f = y$ in (1) and $g = x$ in (2) $\Longrightarrow$

$$\text{Area of } R \;=\; \iint_R 1\,dx\,dy \;=\; -\oint_C y\,dx \;=\; \oint_C x\,dy\,.$$

Example: Let $R = \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ in the positive quadrant $x \geq 0$, $y \geq 0$. Use Green's theorem to evaluate

$$I \;=\; \oint_C [\,(3x - y)\,dx + (x + 2y)\,dy\,]\,,$$

where C is the boundary of R.

$f(x,y) = 3x - y$ and $g(x,y) = x + 2y \Longrightarrow \frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) = 2$, and hence (3) $\Longrightarrow$

$$I = \iint_R 2\,dx\,dy = 2 \times \text{Area of } R = \tfrac{1}{2}\pi ab\,;$$

since the area of the elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ is πab. (To see this, substitute $X = x/a$, $Y = y/b$).

Example: Let R be the region in $(x,\,y)$ plane bounded by the curves $y = x$ and $y = x^2$. Evaluate

$$I \;=\; \oint_C [\,y^3\,dx + (x^3 + 3xy^2)\,dy\,]\,,$$

where C is the boundary of R, (i) directly, (ii) using Green's theorem.

(i) $C = C_1 \bigcup C_2$, where C_1 is $y = x^2$, $x = 0 \to 1$ and C_2 is $y = x$, $x = 1 \to 0$. Then

$$\begin{aligned}
I \;&=\; \int_{C_1} [\,y^3\,dx + (x^3 + 3xy^2)\,dy\,] \;+ \int_{C_2} [\,y^3\,dx + (x^3 + 3xy^2)\,dy\,] \\
&=\; \int_0^1 [\,x^6\,dx + (x^3 + 3x^5)\,2x\,dx\,] \;-\; \int_0^1 [\,x^3\,dx + (x^3 + 3x^3)\,dx\,] \\
&=\; \int_0^1 (\,7x^6 + 2x^4 - 5x^3\,)\,dx \;=\; 1 + \tfrac{2}{5} - \tfrac{5}{4} \;=\; \tfrac{3}{20}\,.
\end{aligned}$$

(ii) $f(x,y) = y^3$ and $g(x,y) = x^3 + 3xy^2 \Longrightarrow \frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) = (3x^2 + 3y^2) - (3y^2) = 3x^2$, and hence (3) $\Longrightarrow$

$$I = \iint_R 3x^2\,dx\,dy \;=\; 3\int_0^1 \left(\int_{x^2}^{x} x^2\,dy \right) dx \;=\; 3\int_0^1 x^2(x - x^2)\,dx \;=\; \tfrac{3}{20}\,.$$

[**N.B. Green's Theorem in the Plane** is a special case of **Stokes Theorem:** For any surface S with normal $\underline{\widehat{\mathbf{n}}}$ bounded by a curve C and any vector field $\underline{\mathbf{F}}$

$$\oint_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \iint_S (\underline{\nabla} \times \underline{\mathbf{F}}) \cdot \underline{\widehat{\mathbf{n}}}\;dS$$

See later for a definition of the symbol $\underline{\nabla}$. This bit is **non-examinable.**]