M.Eng 2.6 Mathematics: Double Integrals

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

If $a = Ih$, then dividing the area under the curve $y = f(x)$ into strips

$$\int_0^a f(x)\,dx = \lim_{I\to\infty\equiv h\to 0} \sum_{i=1}^{I} f(ih)\,h\,.$$

Rectangle $R = [0,a]\times[0,b]$ in the (x,y) plane. If $a = Ih$ and $b = Jk$ then define

$$\begin{aligned}
\iint_R f(x,y)\,dx\,dy &= \lim_{I,\,J\to\infty\equiv h,\,k\to 0} \sum_{i=1}^{I}\sum_{j=1}^{J} f(ih,jk)\,h\,k \\
&= \lim_{I\to\infty} \sum_{i=1}^{I} \left(\lim_{J\to\infty} \sum_{j=1}^{J} f(ih,jk)\,k \right) h \\
&= \lim_{I\to\infty} \sum_{i=1}^{I} \left(\int_0^b f(ih,y)\,dy \right) h \;=\; \int_0^a \left(\int_0^b f(x,y)\,dy \right) dx\,.
\end{aligned}$$

We can therefore do the integrals sequentially, treating x as constant while doing the y-integration. Since the summation order can be reversed, i.e.

$$\sum_{i=1}^{I}\sum_{j=1}^{J} \;\equiv\; \sum_{j=1}^{J}\sum_{i=1}^{I}\,,$$

we have that

$$\iint_R f(x,y)\,dx\,dy \;=\; \int_0^a \left(\int_0^b f(x,y)\,dy \right) dx \;=\; \int_0^b \left(\int_0^a f(x,y)\,dx \right) dy\,.$$

Example: $f(x,y) = x + y$

$$\begin{aligned}
\iint_R (x+y)\,dx\,dy &= \int_0^a \left(\int_0^b (x+y)\,dy \right) dx \;=\; \int_0^a \left[\, xy + \tfrac{1}{2}y^2 \,\right]_{y=0}^{b} dx \\
&= \int_0^a (bx + \tfrac{1}{2}b^2)\,dx \;=\; \left[\, \tfrac{1}{2}bx^2 + \tfrac{1}{2}b^2 x \,\right]_{x=0}^{a} \;=\; \tfrac{1}{2}(a^2 b + ab^2).
\end{aligned}$$

$$\begin{aligned}
\iint_R (x+y)\,dx\,dy &= \int_0^b \left(\int_0^a (x+y)\,dx \right) dy \;=\; \int_0^b \left[\, \tfrac{1}{2}x^2 + xy \,\right]_{x=0}^{a} dy \\
&= \int_0^b (\tfrac{1}{2}a^2 + ay)\,dy \;=\; \left[\, \tfrac{1}{2}a^2 y + \tfrac{1}{2}ay^2 \,\right]_{y=0}^{b} \;=\; \tfrac{1}{2}(a^2 b + ab^2).
\end{aligned}$$

Variable Limits: For non-rectangular domains R the process is similar, but great care must be taken to find the correct limits, especially when changing the order of integration.

Example: $R = x^2 + y^2 \le a^2$ in the positive quadrant $x \ge 0$, $y \ge 0$.

$$\iint_R f(x,y)\,dx\,dy \;=\; \int_0^a \left(\int_0^{(a^2-x^2)^{\frac{1}{2}}} f(x,y)\,dy \right) dx \;=\; \int_0^a \left(\int_0^{(a^2-y^2)^{\frac{1}{2}}} f(x,y)\,dx \right) dy\,.$$

Equivalent notation:

$$\int_0^a \left(\int_0^{(a^2-x^2)^{\frac{1}{2}}} f(x,y)\,dy \right) dx \;\equiv\; \int_0^a dx \int_0^{(a^2-x^2)^{\frac{1}{2}}} f(x,y)\,dy\,.$$

When reversing the order of integration: It's **essential** to draw a diagram:

$$\begin{aligned} \int_0^{\frac{1}{2}} \left(\int_{y-1}^{-y} f(x,y)\,dx \right) dy \;&=\; \iint_R f(x,y)\,dx\,dy \;=\; \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy \\ &=\; \int_{-1}^{-\frac{1}{2}} \left(\int_0^{x+1} f(x,y)\,dy \right) dx + \int_{-\frac{1}{2}}^{0} \left(\int_0^{-x} f(x,y)\,dy \right) dx\,. \end{aligned}$$

$$\begin{aligned} \int_0^1 \left(\int_0^{\cos^{-1} y} y \sec x\,dx \right) dy \;&=\; \iint_R y \sec x\,dx\,dy \\ &=\; \int_0^{\frac{1}{2}\pi} \left(\int_0^{\cos x} y \sec x\,dy \right) dx \;=\; \tfrac{1}{2} \int_0^{\frac{1}{2}\pi} \cos x\,dx \;=\; \tfrac{1}{2}\,. \end{aligned}$$