Changing Variables in Double Integrals: Jacobians

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

Recall: In one dimension if $x = x(t)$ with $a = x(t_a)$ and $b = x(t_b)$, then

$$\int_a^b f(x)\,dx = \int_{t_a}^{t_b} f(x(t))\,\frac{dx}{dt}\,dt\,. \tag{1}$$

In two dimensions one can show that

$$\iint_R f(x,y)\,dx\,dy \quad = \quad \iint_{R'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,, \tag{2}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} \quad = \quad \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{3}$$

is the **Jacobian** of the transformation.

In one dimension $\frac{dx}{dt} = 1/(\frac{dt}{dx})$, but in two dimensions $\frac{\partial x}{\partial u} \neq 1/(\frac{\partial u}{\partial x})$; e.g. polar coordinates

$$x = r\cos\theta, y = r\sin\theta \Longrightarrow r^2 = x^2 + y^2 \qquad \text{then} \qquad \frac{\partial x}{\partial r} = \cos\theta \neq 1/(\frac{\partial r}{\partial x}) = 1/(\cos\theta)\,.$$

This is because θ is being held constant in $\frac{\partial x}{\partial r}$ whereas y is constant in $\frac{\partial r}{\partial x}$. However,

$$\frac{\partial(x,y)}{\partial(u,v)} \quad = \quad 1/\left(\frac{\partial(u,v)}{\partial(x,y)}\right)\,. \tag{4}$$

Proof:

$$\begin{aligned} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} &= \begin{bmatrix} \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix} = I\,. \end{aligned}$$

As $\det(AB) = \det A \det B$ and $\det I = 1$, we have that

$$\frac{\partial(x,y)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = |I| = 1$$

and hence the desired result (4).

Example: $f(x,y) = x + y$ and $R = a^2 \le x^2 + y^2 \le b^2$ in the positive quadrant $x \ge 0,\ y \ge 0$.

Introduce polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, then $R' \equiv [a,b] \times [0, \frac{\pi}{2}]$ in the $(r,\,\theta)$ plane and

$$\tfrac{\partial(x,y)}{\partial(r,\theta)} \equiv \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{5}$$

Hence (2) and (5) yield that

$$\begin{aligned} \iint_R (x+y)\,dx\,dy &= \iint_{R'} r(\cos\theta + \sin\theta)\, r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\int_a^b r^2(\cos\theta + \sin\theta)\, dr \right) d\theta \\ &= \tfrac{1}{3}(b^3 - a^3) \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\, d\theta \;=\; \tfrac{2}{3}(b^3 - a^3)\,. \end{aligned}$$

Example: $f(x,y) = x^2 y^2$ and R is the region bounded by the curves $y = \frac{1}{2}x^2$, $y = x^2$, $y^2 = x$ and $y^2 = 2x$.

Introduce $u = \frac{x^2}{y}$, $v = \frac{y^2}{x} \Longrightarrow uv = xy$, then $R' \equiv [1,2] \times [1,2]$ in the $(u,\,v)$ plane and

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &\equiv 1/\left(\frac{\partial(u,v)}{\partial(x,y)} \right) \equiv 1/\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \\ &= 1/\begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 1/3\,. \end{aligned} \tag{6}$$

Hence (2) and (6) yield that

$$\begin{aligned} \iint_R x^2 y^2\, dx\, dy &= \iint_{R'} u^2 v^2\, \tfrac{1}{3}\, du\, dv \\ &= \tfrac{1}{3} \int_1^2 \left(\int_1^2 u^2 v^2\, du \right) dv \\ &= \left(\tfrac{1}{3}\right)\left(\tfrac{7}{3}\right) \int_1^2 v^2\, dv \;=\; \left(\tfrac{1}{3}\right)\left(\tfrac{7}{3}\right)\left(\tfrac{7}{3}\right) \;=\; \tfrac{49}{27}\,. \end{aligned}$$