M.Eng 2.6 Mathematics: Line Integrals

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

Any curve C joining the points with position vectors $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ can be represented parametrically as

$$\underline{\mathbf{r}}(t) = x(t)\underline{\mathbf{i}} + y(t)\underline{\mathbf{j}} + z(t)\underline{\mathbf{k}} \qquad t_A \leq t \leq t_B \; .$$

where $\underline{\mathbf{r}}(t)$ is the position vector of a general point on the curve and $\underline{\mathbf{r}}(t_A) = \underline{\mathbf{A}}$ and $\underline{\mathbf{r}}(t_B) = \underline{\mathbf{B}}$. As t increases, the rate of change of $\underline{\mathbf{r}}$ is

$$\frac{d\underline{\mathbf{r}}}{dt} = \frac{dx}{dt}\underline{\mathbf{i}} + \frac{dy}{dt}\underline{\mathbf{j}} + \frac{dz}{dt}\underline{\mathbf{k}} \qquad \text{or symbolically} \qquad d\underline{\mathbf{r}} = dx\,\underline{\mathbf{i}} + dy\,\underline{\mathbf{j}} + dz\,\underline{\mathbf{k}} \; .$$

The vector $\frac{d\underline{\mathbf{r}}}{dt}$ points in the direction in which $\underline{\mathbf{r}}$ changes, i.e. along the curve C. Its magnitude indicates the distance moved along the curve due to an infinitesimal change in t. We write

$$\frac{ds}{dt} \equiv \left|\frac{d\underline{\mathbf{r}}}{dt}\right| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}} . \qquad (1)$$

Integrating, we define the **arclength** function, $s(t)$,

$$s(t) = s(t_A) + \int_{t_A}^{t} \left|\frac{d\underline{\mathbf{r}}(\tau)}{d\tau}\right| d\tau \; .$$

The length, L, of the curve between $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ is therefore

$$L = \int_{t_A}^{t_B} \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{\frac{1}{2}} dt \; .$$

Note it is always possible to use arclength to parameterise the curve C. If $t = s$ above, then $\dfrac{d\underline{\mathbf{r}}}{ds}(s)$ is the UNIT tangent to the curve C at $\underline{\mathbf{r}}(s)$. It may also be possible to use $t = x$ as the parameteric variable. For a curve in the $(x,\, y)$ plane, the arclength can then be written

$$L = \int_{x_A}^{x_B} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \; .$$

Example: A circle of radius a

$$\underline{\mathbf{r}}(t) = a\cos(2\pi t)\underline{\mathbf{i}} + a\sin(2\pi t)\underline{\mathbf{j}} \qquad 0 \leq t \, .$$

We note the curve repeats after $t = 1$. Then

$$\frac{d\underline{\mathbf{r}}}{dt}(t) = -2\pi a\sin(2\pi t)\underline{\mathbf{i}} + 2\pi a\cos(2\pi t)\underline{\mathbf{j}} \quad \Longrightarrow \quad \left|\frac{d\underline{\mathbf{r}}}{dt}(t)\right| = 2\pi a \qquad 0 \leq t \, . \qquad (2)$$

Therefore (1) yields that $\frac{ds}{dt} = 2\pi a \Longrightarrow s(t) = 2\pi a t$ and hence an equivalent parametrisation is

$$\underline{\mathbf{r}}^{\star}(s) = a\cos(\frac{s}{a})\underline{\mathbf{i}} + a\sin(\frac{s}{a})\underline{\mathbf{j}} \quad \Longrightarrow \quad \left|\frac{d\underline{\mathbf{r}}^{\star}}{ds}(s)\right| = 1 \qquad 0 \leq s \, .$$

Let C be an orientated curve (has direction) going from $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$ parameterised by $\underline{\mathbf{r}}(s)$, $s_A \le s \le s_B$, where s is arclength. Given a scalar field $g(\underline{\mathbf{r}}) \equiv g(x,y,z)$, we define

$$\int_C g\,ds \equiv \int_{C_{A\to B}} g\,ds \;=\; \int_{s_A}^{s_B} g(\underline{\mathbf{r}}(s))\,ds\,. \qquad (3)$$

Note that $\int_{C_{B\to A}} g\,ds = -\int_{C_{A\to B}} g\,ds$.

If C is a closed curve, i.e. $\underline{\mathbf{B}} = \underline{\mathbf{A}}$, we write $\oint_C g\,ds$.

If C going from $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$ is parameterised by $\underline{\mathbf{r}}(t)$, $t_A \le t \le t_B$ for a general parameter t rather than arclength s, then on noting (1)

$$\int_C g\,ds \;=\; \int_{t_A}^{t_B} g(\underline{\mathbf{r}}(t))\,\frac{ds}{dt}(t)\,dt \;=\; \int_{t_A}^{t_B} g(\underline{\mathbf{r}}(t))\,\left|\frac{d\underline{\mathbf{r}}}{dt}(t)\right|\,dt\,. \qquad (4)$$

Example: $g(\underline{\mathbf{r}}) \equiv g(x,y,z) = 2xy$ and $\underline{\mathbf{r}}(t) = a\cos(2\pi t)\underline{\mathbf{i}} + a\sin(2\pi t)\underline{\mathbf{j}}$, $0 \le t \le \frac{1}{4}$.
(4) and (2) $\Longrightarrow$

$$\begin{aligned}\int_C g\,ds &= 2\pi a\int_0^{\frac{1}{4}} g(\underline{\mathbf{r}}(t))\,dt \;=\; 4\pi a\int_0^{\frac{1}{4}} (a\cos(2\pi t))(a\sin(2\pi t))\,dt\\ &= 2\pi a^3\int_0^{\frac{1}{4}} \sin(4\pi t)\,dt \;=\; a^3\,.\end{aligned}$$

Let C go from $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$ parameterised by $\underline{\mathbf{r}}(s)$, $s_A \le s \le s_B$; s arclength. Given a vector field $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv F_1(\underline{\mathbf{r}})\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\underline{\mathbf{k}}$, we define

$$\begin{aligned}\int_C \underline{\mathbf{F}}\,.\,d\underline{\mathbf{r}} &= \int_C (F_1dx + F_2dy + F_3dz)\\ &= \int_C \left(F_1\frac{dx}{ds} + F_2\frac{dy}{ds} + F_3\frac{dz}{ds}\right)\,ds\\ &= \int_{s_A}^{s_B}\left[\underline{\mathbf{F}}(\underline{\mathbf{r}}(s))\,.\,\frac{d\underline{\mathbf{r}}}{ds}(s)\right]\,ds\,. \qquad (5)\end{aligned}$$

$\underline{\mathbf{F}}(\underline{\mathbf{r}}(s))\,.\,\frac{d\underline{\mathbf{r}}}{ds}(s)$ is the component of $\underline{\mathbf{F}}(\underline{\mathbf{r}}(s))$ in the direction of the tangent to the curve at $\underline{\mathbf{r}}(s)$.

Therefore $\int_C \underline{\mathbf{F}}\,.\,d\underline{\mathbf{r}}$ is the integral of the **tangential component** of $\underline{\mathbf{F}}$ along C.

Physical Interpretation: It is the **work done** by the force $\underline{\mathbf{F}}$ in moving a particle from $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$ along the path C.

If C is a closed curve, we write $\oint_C \underline{\mathbf{F}}\,.\,d\underline{\mathbf{r}}$.

If C going from $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$ is parameterised by $\underline{\mathbf{r}}(t)$, $t_A \le t \le t_B$; a general parameter t, then

$$\int_C \underline{\mathbf{F}}\,.\,d\underline{\mathbf{r}} \;=\; \int_{t_A}^{t_B}\left[\underline{\mathbf{F}}(\underline{\mathbf{r}}(t))\,.\,\frac{d\underline{\mathbf{r}}}{dt}(t)\right]\,dt\,. \qquad (6)$$

If t denotes time, and $\underline{\mathbf{r}}(t)$ is the position vector of a body, then $\frac{d\underline{\mathbf{r}}}{dt}$ is its velocity, and $\underline{\mathbf{F}}\cdot\frac{d\underline{\mathbf{r}}}{dt}$ is the **rate of working** of the force $\underline{\mathbf{F}}$.