M.Eng. 2.6 Mathematics: More on ODEs with Constant Coefficients.

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

Let $\underline{\mathbf{x}}(t)$ be an n-dimensional vector which varies with time, and suppose that

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} \qquad \text{where } A \text{ is a constant, real } n \times n \text{ matrix.} \tag{1}$$

Recall from earlier in the course how to solve this problem: If A has n eigenvectors, $\underline{\mathbf{e}}_1$, $\underline{\mathbf{e}}_2$ $\ldots \underline{\mathbf{e}}_n$, with corresponding eigenvalues λ_1, λ_2 $\ldots \lambda_n$, so that $A\underline{\mathbf{e}}_k = \lambda_k \underline{\mathbf{e}}_k$ for $k = 1 \ldots n$, then the **General Solution** to (1) can be written

$$\underline{\mathbf{x}} = c_1 \underline{\mathbf{e}}_1 e^{\lambda_1 t} + c_2 \underline{\mathbf{e}}_2 e^{\lambda_2 t} + \ldots + c_n \underline{\mathbf{e}}_n e^{\lambda_n t} \ ,$$

where c_1, c_2 $\ldots c_n$ are arbitrary constants. If we know the value of $\underline{\mathbf{x}}$ at, say, $t = 0$, then we can find the constants and determine the unique solution.

This procedure is fairly straightforward if the eigenvalues are all real. If, however, two or more of the λ_k are complex, then so will be the eigenvectors and constants. The analysis goes through unchanged, but we may want to rewrite $e^{(\alpha+i\beta)t}$ as $e^{\alpha t}(\cos\beta t + i \sin\beta t)$.

Why is equation (1) important? Because **every** set of linear differential equations with constant coefficients can be written in that form, by defining new unknown variables. For example, the charge $Q(t)$ flowing in an electric circuit of capacitance C, resistance R and inductance L obeys the equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = 0 \ . \tag{2}$$

If we want, we can rewrite this equation as

$$\dot{\underline{\mathbf{x}}} = A\underline{\mathbf{x}} \qquad \text{where} \quad \underline{\mathbf{x}} = \begin{pmatrix} Q \\ \dot{Q} \end{pmatrix} \qquad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{pmatrix} \ .$$

We can then find the eigenvalues in the usual way, setting $\det(A - \lambda I) = 0$.

$$0 = \begin{vmatrix} -\lambda & 1 \\ -\frac{1}{LC} & -\frac{R}{L} - \lambda \end{vmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} \ .$$

Solving this quadratic gives the eigenvalues λ_1 and λ_2 for which the eigenvectors are $(1,\, \lambda_1)^T$ and $(1,\, \lambda_2)^T$ respectively. Although in this case it is easier to solve (2) directly, this may not be the case for more complicated circuits.

Another important example of (1) occurs in the study of vibrations and stability of structures. Consider a mechanical system with n degrees of freedom which we measure with $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_n)^T$. If the system is disturbed slightly from equilibrium, its motion is governed by an equation like

$$M\ddot{\underline{\mathbf{x}}} + K\underline{\mathbf{x}} = 0 \qquad \text{where } M \text{ and } K \text{ are symmetric, constant } n \times n \text{ matrices.} \tag{3}$$

By introducing new variables for $\dot{x}_1$ etc, we could write (3) in the form of (1). Equivalently, we can look for solutions of (3) behaving like $\underline{\mathbf{x}} = \underline{\mathbf{e}}\, e^{i\omega t}$ or, if you hate complex numbers, $\underline{\mathbf{x}} = \underline{\mathbf{e}} \cos \omega t$. Here $\underline{\mathbf{e}}$ is a constant vector, called a **normal mode** of vibration. The corresponding value of ω is called a **normal frequency** [*c.f.* "characteristic values" in your Vibrations course.] Substituting in (3), we see that it is a solution provided

$$K\underline{\mathbf{e}} = \omega^2 M\underline{\mathbf{e}} \qquad \text{or} \quad K\underline{\mathbf{e}} = \omega^2\underline{\mathbf{e}} \qquad \text{taking } M = I \text{ for simplicity.} \tag{4}$$

Therefore if M is the identity matrix, we need $\underline{\mathbf{e}}$ to be an eigenvector of K with corresponding eigenvalue λ and $\omega = \pm\sqrt{\lambda}$.

As the matrix K is symmetric the eigenvalue λ is real. If $\lambda < 0$, then ω is imaginary and the equilibrium is **unstable**, whereas if all the eigenvalues are positive it is **stable**. The frequencies, and modes of vibration are then given by the possible values of ω and the eigenvectors $\underline{\mathbf{e}}$. The general motion of the system is the superposition of all the normal modes

$$\underline{\mathbf{x}} = \underline{\mathbf{e}}_1(a_1 \cos\sqrt{\lambda_1}t + b_1 \sin\sqrt{\lambda_1}t) + \underline{\mathbf{e}}_2(a_2 \cos\sqrt{\lambda_2}t + b_2 \sin\sqrt{\lambda_2}t) + \ldots \tag{5}$$

Example: Two heavy masses are connected by three light stretched strings to each other and to two fixed points. Their displacements perpendicular to the strings are $x_1(t)$ and $x_2(t)$. Discuss the resulting equation of motion

$$\ddot{\underline{\mathbf{x}}} + K\underline{\mathbf{x}} = 0 \qquad \text{where} \quad \underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{and} \quad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{6}$$

Writing $\underline{\mathbf{y}} = (x_1,\, \dot{x}_1,\, x_2,\, \dot{x}_2)^T$ find the 4×4 matrix A such that $\dot{\underline{\mathbf{y}}} = A\underline{\mathbf{y}}$.
Find the eigenvectors and eigenvalues of K and hence the general motion of the system.