M.Eng 2.6 Mathematics: Problem Sheet 5: Double and Line Integrals

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

1. Evaluate (a) $\displaystyle\int_1^b \left[\int_0^a x\,y^{-1}\,dx\right] dy$, (b) $\displaystyle\int_0^a \left[\int_0^{y^2} x\,y\,dx\right] dy$.

2. By inverting the order of integration, evaluate $\displaystyle\int_0^1 \left[\int_y^1 \frac{ye^x}{x}\,dx\right] dy$.

3. A fluid has the velocity field $\underline{\mathbf{v}} = 2xz\underline{\mathbf{i}} + 2yz\underline{\mathbf{j}} + (1 + x^2 + y^2)\underline{\mathbf{k}}$. Show that the volume V crossing an area R in the (x, y) plane per second is given by

$$V = \iint_R (1 + x^2 + y^2)\,dx\,dy\,.$$

   Calculate V when R is the triangle with vertices $(0, 0)$, $(1, 1)$ and $(1, 2)$.

4. Evaluate $\displaystyle\iint_R (a^2 - x^2)\,dx\,dy$, where R is the area within the disc $x^2 + y^2 \le a^2$, by transforming to plane polar coordinates (r, θ).

5. Use the transformation $u = 2x - y$, $v = y$ to evaluate the integral $\displaystyle\iint_R \cos[(2x - y)^2]\,dx\,dy$, where R is the triangle with vertices $(0, 0)$, $(0, -1)$ and $(1, 1)$.

6. Evaluate $\displaystyle\int_C 3x\,ds$ from $(0, 0)$ to $(1, 1)$ along the paths (a) $y = x$, and (b) $y = x^2$.

7. Evaluate $\displaystyle\int_C \left[(x^2 + y^2)\,dx - 2xy\,dy\right]$ from $(0, 0)$ to $(1, 1)$ along the paths (a) $y = x$, (b) $y = \sqrt{x}$, (c) $y = x^2$.

8. Evaluate $\displaystyle\oint_C (x\,dy - y\,dx)$, where C is the closed curve $x = \cos^3 t$, $y = \sin^3 t$.
   [N.B. $\int_0^{\pi/2} \sin^2 t \cos^2 t\,dt = \frac{1}{16}\pi$]

9. Evaluate $\displaystyle\int_C (y\cos x\,dx + \sin x\,dy)$ from $(0, 0)$ to $(\frac{\pi}{2}, \frac{\pi}{2})$ along the two paths with straight line segments (a) $x = 0$ and $y = \frac{\pi}{2}$, (b) $y = 0$ and $x = \frac{\pi}{2}$. Why are the results the same ?

10. If $\underline{\mathbf{F}} = (3x^2 + 6y)\underline{\mathbf{i}} - 14yz\underline{\mathbf{j}} + 20xz^2\underline{\mathbf{k}}$, evaluate $\displaystyle\int_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t$, $y = t^2$, $z = t^3$.

11. If $\underline{\mathbf{F}} = (y^2 \cos x + z^3)\underline{\mathbf{i}} + (2y \sin x - 4)\underline{\mathbf{j}} + (3xz^2 + 2)\underline{\mathbf{k}}$, show that $\underline{\mathbf{F}}$ is a conservative vector field. Find the corresponding scalar potential ϕ and evaluate the work done by $\underline{\mathbf{F}}$ in moving from from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$.

12. Evaluate $\oint_C \left[(x^2 - 2xy)\,dx + (x^2y + 3)\,dy\right]$ around the boundary of the region bounded by $y^2 = 8x$ and $x = 2$ in the $(x,\, y)$ plane (a) directly, (b) using Green's theorem.

13. With $f = x^2/(x+y)$ and $g = -y^2/(x+y)$, evaluate by means of Green's theorem the integral

$$\int\!\!\int_R \frac{x^2 + y^2}{(x+y)^2}\,dx\,dy,$$

where R is the region $x^2 + y^2 \leq a^2$, $x \geq 0$ and $y \geq 0$.

ANSWERS

1. (a) $\frac{1}{2}a^2 \ln b$, (b) $\frac{1}{12}a^6$.
2. $\frac{1}{2}$.
3. $\frac{4}{3}$.
4. $\frac{3}{4}\pi a^4$.
5. $\frac{1}{2}\sin(1)$
6. (a) $\frac{3\sqrt{2}}{2}$, (b) $\frac{1}{4}(5\sqrt{5} - 1)$.
7. (a) 0, (b) $\frac{1}{3}$, (c) $-\frac{4}{15}$.
8. $\frac{3\pi}{4}$.
9. $\frac{\pi}{2}$.
10. 5.
11. $\phi = y^2 \sin x + xz^3 - 4y + 2z + c$, $15 + 4\pi$.
12. $\frac{128}{5}$.
13. $\frac{1}{2}a^2$.