M.Eng 2.6 Problem Sheet 6: Vector Calculus

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

Throughout this sheet we use the notation $\underline{\mathbf{r}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$ and $r = |\underline{\mathbf{r}}|$.

1. Find the directional derivative of $\phi(x,y,z) = 4xz^3 - 3x^2y^2z$ in the direction $2\underline{\mathbf{i}} - 3\underline{\mathbf{j}} + 6\underline{\mathbf{k}}$ at the point $(2,-1,2)$.
2. Obtain the gradients of the following scalar fields:-
   (i) y, (ii) $x^3 + y^3 + z^3$, (iii) $\underline{\mathbf{r}} \cdot \underline{\nabla}(x+y+z)$, (iv) r^n.
3. Find the equation of the tangent plane to the surface $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$.
4. Show that $(\underline{\mathbf{F}} \cdot \underline{\nabla})\underline{\mathbf{r}} = \underline{\mathbf{F}}$ for any vector field $\underline{\mathbf{F}}$.
5. If $\underline{\mathbf{a}}$ is a constant vector and $\underline{\mathbf{v}} = (\underline{\mathbf{a}} \cdot \underline{\mathbf{r}})\underline{\mathbf{r}}$, find $\underline{\nabla} \cdot \underline{\mathbf{v}}$ and $\underline{\nabla} \times \underline{\mathbf{v}}$.
6. Obtain the curls of the following vector fields:-
   (i) $x\,\underline{\mathbf{i}}$, (ii) $y\,\underline{\mathbf{i}}$, (iii) $\underline{\mathbf{r}}$, (iv) $f(r)\,\underline{\mathbf{r}}$, (v) $(x\,\underline{\mathbf{i}} - y\,\underline{\mathbf{j}})/(x+y)$.
7. If f is an arbitrary scalar field and $\underline{\mathbf{F}}$, $\underline{\mathbf{G}}$ are arbitrary vector fields, verify the identities:-
   (i) $\underline{\nabla} \times (\underline{\nabla} f) = \underline{0}$, (ii) $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) = 0$, (iii) $\underline{\nabla} \cdot (\underline{\mathbf{F}} \times \underline{\mathbf{G}}) = \underline{\mathbf{G}} \cdot (\underline{\nabla} \times \underline{\mathbf{F}}) - \underline{\mathbf{F}} \cdot (\underline{\nabla} \times \underline{\mathbf{G}})$.
8. If $\underline{\mathbf{F}} = x^2y\,\underline{\mathbf{i}} - 2xz\,\underline{\mathbf{j}} + 2yz\,\underline{\mathbf{k}}$, show that $\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{F}}) = -\nabla^2\underline{\mathbf{F}} + \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{F}})$,
   where $\nabla^2(F_1\,\underline{\mathbf{i}} + F_2\,\underline{\mathbf{j}} + F_3\,\underline{\mathbf{k}}) \equiv (\nabla^2 F_1)\,\underline{\mathbf{i}} + (\nabla^2 F_2)\,\underline{\mathbf{j}} + (\nabla^2 F_3)\,\underline{\mathbf{k}}$.
9. Prove that $\nabla^2 f(r) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$. Hence show that $f(r) = A + Br^{-1}$, where A and B are arbitrary constants, satisfies $\nabla^2 f = 0$.

ANSWERS

1. 376/7.
2. (i) $\underline{\mathbf{j}}$, (ii) $3(x^2\underline{\mathbf{i}} + y^2\underline{\mathbf{j}} + z^2\underline{\mathbf{k}})$, (iii) $\underline{\mathbf{i}} + \underline{\mathbf{j}} + \underline{\mathbf{k}}$, (iv) $nr^{n-2}\underline{\mathbf{r}}$.
3. $-2x + y + 3z = 1$.
4. —
5. $4(\underline{\mathbf{a}} \cdot \underline{\mathbf{r}})$ and $\underline{\mathbf{a}} \times \underline{\mathbf{r}}$.
6. (i) $\underline{0}$, (ii)$-\underline{\mathbf{k}}$, (iii) $\underline{0}$, (iv) $\underline{0}$, (v) $(x+y)^{-1}\underline{\mathbf{k}}$.
7. —
8. $2(x+1)\underline{\mathbf{j}}$.
9. —