This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

1. Find the Fourier series of the function $f(x)$ where

$$f(x) = 0 \qquad \text{for} \quad -\pi < x < 0 \qquad \text{and} \quad f(x) = 1 \qquad \text{for} \quad 0 < x < \pi \ .$$

What is the value of the series at $x = 0$, where $f(x)$ is discontinuous?

{*Answer:* $f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{\text{n odd}} \sin nx/n$}

2. Show that if $-\pi < x < \pi$,

$$x^2 = \tfrac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \ ,$$

By differentiating this series, infer the Fourier series of x in the same interval. By integrating the series, and using the series for x you have just found, find a similar series for x^3.

{*Answer:* $x^3 = \sum (-1)^n \left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right] \sin nx$}

3. Using Parseval's theorem for the series for x, x^2 and x^3 calculated in question 2, show that

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ , \qquad (b) \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \ , \qquad (c) \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \ .$$

4. Consider the $2L$-periodic "square pulse" function $f(x) = 1$ for $0 \leqslant |x| < 1$, $f(x) = 0$ for $1 < |x| < L$. Calculate the Fourier coefficients a_n. Defining $k_n = n\pi/L$, and $F_n = La_n$, show that $F_n = 2\sin k_n/k_n$.

Now let $L \to \infty$ and define $F(k)$, the **Fourier Transform** of $f(x)$, by

$$F(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \ .$$

Show that $F(k) = 2\sin k/k$.

[This illustrates the connection between Fourier Series and Fourier Transforms.]