This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

**1.** Find all vectors $\underline{\mathbf{x}}$ which obey the differential equation

$$\frac{d\underline{\mathbf{x}}}{dt} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix} \underline{\mathbf{x}} \,,$$

in terms of the eigenvalues and eigenvectors of the matrix.
At $t = 0$, $\underline{\mathbf{x}} = (1,\, 2,\, -7)^T$. Find $x(t)$ and describe what happens to $\underline{\mathbf{x}}$ as $t \to \infty$.

{*Answer:* $\underline{\mathbf{x}} = e^{-t}(1, 2, -7)^T$}
[For later: Would a computer get this right? What is the effect of small rounding errors?]

**2.** The vector $\underline{\mathbf{x}}$ satisfies the differential equation

$$t\frac{d\underline{\mathbf{x}}}{dt} = A\underline{\mathbf{x}} \qquad \text{for a constant matrix } A.$$

Show that there exist solutions of the form $\underline{\mathbf{x}} = t^{\lambda}\underline{\mathbf{a}}$, where $\lambda$ is a constant and $\underline{\mathbf{a}}$ is a constant vector, provided $A\underline{\mathbf{a}} = \lambda\underline{\mathbf{a}}$.

Use this technique to solve the problem

$$t\frac{d\underline{\mathbf{x}}}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \underline{\mathbf{x}} \,.$$

{*Answer:* $\underline{\mathbf{x}} = c_1 t^2 (1, 3)^T + c_2 t^4 (1, 1)^T$}

**3.** Writing $'$ for $d/dt$, show that the scalar equation

$$y''' + 5y'' + 6y' = 0$$

can be written in the form $\underline{\mathbf{x}}' = A\underline{\mathbf{x}}$, for a suitable $2 \times 2$ matrix $A$ and vector $\underline{\mathbf{x}}$. Calculate the eigenvalues and eigenvectors of $A$, and hence find $y(t)$ when $y(0) = 2$, $y'(0) = -2$ and $y''(0) = 4$.

{*Answer:* $y = 1 + e^{-2t}$}

**4.** Three equal masses are connected in a straight line to two fixed points using 4 identical springs. The masses can move normal to the springs with displacements $x_1$, $x_2$ and $x_3$. Writing $\underline{\mathbf{x}} = (x_1,\, x_2,\, x_3)^T$, the equation which governs the vibration of the system is, in suitable units,

$$\frac{d^2\underline{\mathbf{x}}}{dt^2} + A\underline{\mathbf{x}} = 0 \qquad \text{where} \quad A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

Show that a possible mode of vibration is $(x_1,\, x_2,\, x_3) = (1,\, 0,\, -1)\, e^{i\omega t}$, and show that $\omega = \sqrt{2}$. Find the other two frequencies and corresponding eigenvectors.

{*Answer:* $\omega = (2 \pm \sqrt{2})^{1/2}$, eigenvectors $(1, \mp\sqrt{2}, 1)^T$.}