This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

1. The function $u(x, t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{in} \quad 0 < x < L, \quad t > 0$$

with the boundary conditons

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 \qquad \text{and} \quad u(x, 0) = f(x) \ .$$

Show that its solution is

$$u(x, t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 t/L^2} \ ,$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx \ .$$

Hence find the solution when

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 < x < \frac{1}{2}L \\ 0 & \text{for} \quad \frac{1}{2}L < x < L \ . \end{cases}$$

What happens as $t \to \infty$?

2. A metal bar of length π and thermal diffusivity k is initially at room temperature, T_0. Its ends are suddenly cooled, so that subsequently the temperature distribution along the bar, $u(x, t)$, is the solution to the problem

$$u_t = k u_{xx} \qquad \text{in} \quad 0 < x < \pi, \ t > 0$$

together with the boundary conditions

$$u(0, t) = u(\pi, t) = 0 \qquad \text{and} \quad u(x, 0) = T_0 \ .$$

Show that the solution of this problem is

$$u(x, t) = \frac{4T_0}{\pi} \sum_{n \ odd}^{\infty} \exp[-n^2 kt] \frac{\sin nx}{n} \ .$$

3. A heat-releasing chemical reaction takes place in a long pipe with insulating walls and fixed temperature at each end. The reaction proceeds at a rate which is proportional to the temperature, so that the temperature distribution $u(x,\, t)$ obeys the equation

$$u_t = ru + ku_{xx} \qquad \text{in} \quad 0 < x < L \ , \quad t > 0$$

where r and k are known constants. The boundary conditions are

$$u(0, t) = u(L,\, t) = 0 \qquad \text{and} \quad u(x, 0) = T(x) \ ,$$

for known $T(x)$. Using separation of variables, show that the temperature is given by

$$u(x,\, t) = \sum_{n=1}^{\infty} b_n \exp(\lambda_n t) \sin\left(\frac{n\pi x}{L}\right)$$

where

$$\lambda_n = r - \frac{n^2\pi^2 k}{L^2} \qquad \text{and} \quad b_n = \frac{2}{L}\int_0^L T(x) \sin\left(\frac{n\pi x}{L}\right) dx \ .$$

Under what conditions will the temperature increase without limit, leading to an explosion?

4. (A question from the 1997 exam:) $u(x,\, y)$ obeys Laplace's equation in a square:

$$u_{xx} + u_{yy} = 0 \qquad \text{in} \quad 0 < x < \pi, \quad 0 < y < \pi \ ,$$

with the boundary conditions

$$u(x,\, 0) = u(x,\, \pi) = u(0,\, y) = 0 \qquad \text{and} \quad u(\pi,\, y) = \sin^3 y \ .$$

Use the method of separation of variables to find the solution.
[Hint: $\sin 3y = 3\sin y - 4\sin^3 y$.]

5. In terms of polar coordinates $(r,\, \theta)$, Laplace's equation for $u(r,\, \theta)$ is

$$\nabla^2 u \equiv u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = 0 \ .$$

Show that the separable solutions $u = f(r)g(\theta)$ which are 2π-periodic in θ and finite at $r = 0$ are $u = A_n r^n \cos n\theta$ and $u = B_n r^n \sin n\theta$ where n is an integer and A_n and B_n are constants.

Using the formula from the previous question, deduce that the problem

$$\nabla^2 u = 0 \qquad \text{in} \quad 0 < r < 1, \quad 0 < \theta < 2\pi \ , \qquad \text{with} \quad u(1, \theta) = \sin^3 \theta \ ,$$

has the solution

$$u(r,\, \theta) = \tfrac{3}{4} r \sin\theta - \tfrac{1}{4} r^3 \sin 3\theta \ .$$