This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/MEng26

1. On the grid $(nh,\, jk)$, for integers n and j, derive the finite difference scheme

$$U_n^{j+1} = rU_{n-1}^j + (1-2r)U_n^j + rU_{n+1}^j\ , \qquad \text{where} \quad r = Dk/h^2\ ,$$

for the heat equation for $u(x,\, t)$,

$$u_t = Du_{xx} \qquad \text{with} \quad u(x,0) = e^{ipx}$$

for some constants D and p. [We take $u(x,\, t)$ to be complex for simplicity.]
Look for a solution

$$U_n^j = (\lambda)^j e^{in\xi} \qquad \text{where} \quad \xi = ph$$

and show that such a solution exists provided

$$\lambda = 1 - 2r(1-\cos\xi)\ .$$

By considering $\xi = \pi$, show that it is possible for $|\lambda| > 1$ if $r > \frac{1}{2}$. How does the solution U_n^j behave as $j \to \infty$ in that case?

2. The truncation error of the Crank-Nicolson scheme for the equation $u_t = u_{xx}$ is defined by

$$E_n^j = \frac{U_n^{j+1} - U_n^j}{k} - \left(\frac{U_{n-1}^j - 2U_n^j + U_{n+1}^j}{2h^2}\right) - \left(\frac{U_{n-1}^{j+1} - 2U_n^{j+1} + U_{n+1}^{j+1}}{2h^2}\right)\ .$$

Show that $E_n^j = O(k^2,\, h^2)$.

3. The function $u(x,\, y)$ obeys Laplace's equation in a rectangle

$$u_{xx} + u_{yy} = 0 \qquad \text{in} \quad 0 < x < 1, \qquad 0 < y < 2\ ,$$

with the boundary conditions

$$u(0,\, y) = 0\ , \qquad u(1,\, y) = 4\ , \qquad u(x,\, 0) = 0\ , \qquad \text{and} \quad u(x,\, 2) = 4\ .$$

Formulate a numerical scheme to solve this problem on a square grid of size h where $Nh = 1$, writing the equations in the form $A\mathbf{v} = \mathbf{b}$. Define the matrix A, the unknown vector $\mathbf{v}$ and the known vector $\mathbf{b}$ precisely.
When $h = \frac{1}{2}$, obtain three simultaneous equations in the unknowns $v_1 = u(h,\, h)$, $v_2 = u(h,\, 2h)$ and $v_3 = u(h,\, 3h)$.
Set up the Jacobi and Gauss-Seidel iterative schemes to solve this problem and perform two iterations of each method. Which method seems to be converging faster?