

## M.Eng. 2.6 Mathematics: Trapezium and Runge-Kutta methods.

This sheet can be found on the Web: <http://www.ma.ic.ac.uk/~ajm8/MEng26>

Euler's method is simple to use, but is not particularly accurate. Thus to keep our error satisfactorily low we may have to choose a very small value of  $h$ . Let's see if we can find a more accurate method which involves only a little more calculation.

An obvious generalisation of Euler's method is

$$y_{n+1} = y_n + h [c_1 f(x_n, y_n) + c_2 f(x_{n+1}, y_{n+1})] ,$$

where  $c_1$  and  $c_2$  are constants which we can choose. In Euler's method  $c_1 = 1$  and  $c_2 = 0$ .

We define the **local error** or **truncation error**,  $E_n$ , at the point  $x = x_n$  to be the amount by which the exact solution fails to satisfy our approximate equation, so that

$$\begin{aligned} E_n &\equiv y(x_{n+1}) - y(x_n) - h [c_1 f(x_n, y(x_n)) + c_2 f(x_{n+1}, y(x_{n+1}))] \\ &= y(x_{n+1}) - y(x_n) - h [c_1 y'(x_n) + c_2 y'(x_{n+1})] , \end{aligned}$$

since  $y' = f(x, y)$ . We now expand  $y(x_{n+1})$  and  $y'(x_{n+1})$  as Taylor series, recalling that  $x_{n+1} = x_n + h$ , so that

$$\begin{aligned} E_n &= y(x_n) + hy'(x_n) + \frac{1}{2}h^2y''(x_n) + \frac{1}{6}h^3y'''(x_n) + \dots \\ &\quad - y(x_n) - hc_1y'(x_n) \\ &\quad - hc_2 [y'(x_n) + hy''(x_n) + \frac{1}{2}h^2y'''(x_n) + \dots] \\ &= [1 - c_1 - c_2]hy'(x_n) + [\frac{1}{2} - c_2]h^2y''(x_n) + [\frac{1}{6} - \frac{1}{2}c_2]h^3y'''(x_n) + \dots \end{aligned}$$

We see therefore that for  $c_1 = 1, c_2 = 0$  (Euler's method)  $E_n = \frac{1}{2}h^2y''(x_n) + O(h^3)$ . Euler's method is **first order**. However, if  $c_2 = \frac{1}{2}$  and  $c_1 = \frac{1}{2}$  then  $E_n = -\frac{1}{12}h^3y'''(x_n) + O(h^4)$ . This more accurate, **second order** method is known as **The Trapezium Method**:

$$y_{n+1} - y_n = \frac{1}{2}h [f(x_n, y_n) + f(x_{n+1}, y_{n+1})] . \quad (1)$$

Let's try this on the same numerical example we used before:

$$y' = 2xy \quad \text{with} \quad y(0) = 1 \quad \text{using} \quad h = \frac{1}{4} . \quad (2)$$

We have  $x_0 = 0, y_0 = 1, x_1 = \frac{1}{4}$  and  $f(x, y) = 2xy$ . Using (1) with  $n = 0$ ,

$$y_1 - y_0 = \frac{1}{8} [2x_0y_0 + 2x_1y_1] \quad \text{or} \quad y_1 - 1 = \frac{1}{8} [0 + \frac{1}{2}y_1] ,$$

giving  $y_1 = \frac{16}{15}$ . Now put  $n = 1$  in (1), to obtain

$$y_2 - \frac{16}{15} = \frac{1}{8} [2(\frac{1}{4})(\frac{16}{15}) + 2(\frac{1}{2})y_2] ,$$

giving  $y_2 = \frac{136}{105} \simeq 1.295$ . Now the exact solution to (2) is  $y = e^{x^2}$ , so that  $y(x_2) = e^{0.25} \simeq 1.284$ . Euler's method gave the result  $y_2 = \frac{9}{8} \simeq 1.125$  which is clearly less accurate.

Unfortunately, the Trapezium method suffers from a severe disadvantage, namely that it is an **implicit** method. It cannot be written in the form  $y_{n+1} = (\text{something calculable})$ . You may have noticed that we had to solve an equation to find the value of  $y_{n+1}$  given  $y_n$ . This wasn't so bad for a **linear** problem, but consider, say,

$$y' = y^{10} \quad \text{with } y(0) = 1 \quad \text{using } h = 1 .$$

Applying the Trapezium method to this problem gives for the first step,

$$y_1 = y_0 + \frac{1}{2} [y_0^{10} + y_1^{10}] \quad \text{or } y_1^{10} - 2y_1 + 3 = 0 .$$

The value of  $y_1$  is given **implicitly** by this equation. We still have work to do to find it, and there may be more than one solution or, as in this case, no solution! Despite being **second order**, the method is not so straightforward to implement.

### Second Order Runge-Kutta methods:

There is a large family of methods which, while having an  $O(h^3)$  local error, are nevertheless **explicit**. The basic idea is first to use Euler's method (or similar) to provide a first estimate of the solution at some other point, and then to use this estimate to obtain an improved version. For this course you need only meet the two most popular methods below:

$$\left. \begin{aligned} y_{n+1}^* &= y_n + hf(x_n, y_n) \\ y_{n+1} &= y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)] \end{aligned} \right\} \quad (3)$$

and

$$\left. \begin{aligned} y_{n+1}^* &= y_n + \frac{1}{2}hf(x_n, y_n) \\ y_{n+1} &= y_n + hf(x_n + \frac{1}{2}h, y_{n+1}^*) \end{aligned} \right\} \quad (4)$$

In these methods  $y_n^*$  is not part of the solution and merely calculated on the way. In (3),  $y_{n+1}^*$  is an approximation to  $y_{n+1}$  obtained by Euler's method, which is then used in an approximate Trapezium method. In (4),  $y_{n+1}^*$  is an approximation to  $y(x_n + \frac{1}{2}h)$ . Convince yourself that these two methods are **explicit** i.e.  $y_{n+1}$  follows by direct calculation from  $y_n$ .

These two methods are shown to be second order on Problem Sheet 2, along with a few examples and a simple PASCAL program. There exist more complicated Runge-Kutta methods of higher accuracy, but we will not discuss those here.