# M1M1: Problem Sheet 8: Complex numbers

**1.** Put the following complex numbers into standard form, i.e., in the form $x + iy$ for some real $x$ and $y$:

$$(a)\ \frac{1}{2+3i};\quad (b)\ \frac{1}{2-3i};\quad (c)\ \frac{1-i}{1+i};\quad (d)\ \frac{1+i}{1-i};\quad (e)\ \frac{1}{i^5};$$

$$(f)\ \frac{(1+i)(2+i)(3+i)}{(1-i)};\quad (g)\ \frac{1}{1+\sqrt{3}i};\quad (h)\ \sqrt{5+12i}.$$

**2.** Let $z_1 = -1+2i$ and $z_2 = 3-2i$. Find the standard form of the following complex numbers:

$$(a)\ 2z_1 - 3z_2;\quad (b)\ z_1 z_2;\quad (c)\ \frac{z_1^2}{z_2};\quad (d)\ |z_1^2 z_2|.$$

Now let $z_1 = 1-3i$ and $z_2 = 3-2i$. Find the standard form of the following complex numbers:

$$(a)\ |z_1 + z_2|;\quad (b)\ |z_2| z_1;\quad (c)\ z_1 + |z_1|;\quad (d)\ \left|\frac{z_1}{z_2}\right|.$$

**3.** Write the following complex numbers in standard form:

$$(a)\ 2e^{i\pi/2};\quad (b)\ 3e^{-i\pi};\quad (c)\ 2e^{-i\pi/2};\quad (d)\ 3e^{i\pi/4};\quad (e)\ 2e^{i\pi/6}.$$

Write the following complex numbers in polar form:

$$(a)\ i;\quad (b)\ \frac{1+i}{\sqrt{2}};\quad (c)\ -1+i\sqrt{3};\quad (d)\ 6+8i;\quad (e)\ -1.$$

**4.** By expressing $-1+i$ in polar form (i.e., in the form $re^{i\theta}$), find the standard form of the number $(-1+i)^{-8}$.

**5.** Show that

$$\frac{1+\cos\theta + i\sin\theta}{1-\cos\theta + i\sin\theta} = \cot\left(\frac{\theta}{2}\right) e^{i(\theta-\pi/2)}.$$

**6.** Given that $2+i$ is a solution of the equation

$$z^4 - 2z^3 - z^2 + 2z + 10 = 0,$$

find the other solutions.

**7.** If $z = e^{i\theta}$, show that

$$\cos(n\theta) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).$$

Hence, show that

$$\cos^6\theta = \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}.$$

**8.** Use the fact that

$$\cos(n\theta) = \mathrm{Re}[e^{in\theta}]$$

to establish the identity

$$1 + \cos(\theta) + \frac{\cos(2\theta)}{2!} + \frac{\cos(3\theta)}{3!} + \frac{\cos(4\theta)}{4!} + \cdots = e^{\cos(\theta)}\cos(\sin(\theta)).$$

**9.** Sketch the following curves in the complex plane:

$$(a)\ |z-i| = |z-1|; \quad (b)\ |z-i| = 2; \quad (c)\ \mathrm{Re}[z^2] = 1;$$

$$(d)\ z\bar{z} = 1; \quad (e)\ \arg\left[\frac{z+1}{z-1}\right] = \pm\frac{\pi}{2}, \quad z \neq \pm 1$$

**10.** By treating it as a quadratic, find the roots of the equation

$$z^{2i} + z^i + 1 = 0.$$

**11.** Three roots of a polynomial equation of degree 5 with real coefficients are $1, i \pm 1$. Find the equation.

**12.** Find all the values of $\log(1+i)$ and $\log\left[(1+i)^{1/i}\right]$.

**13.** Find all complex solutions to the following equations:

$$(a)\ e^z = -2; \quad (b)\ z^7 = -1; \quad (c)\ \cos z = \sqrt{2}.$$

**13$\frac{1}{2}$.** Infer the radius of convergence of the Maclaurin series of the function $1/(2+e^x)$.

**14.** Use de Moivre's theorem to show that

$$\cos(5\theta) = 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta),$$
$$\sin(5\theta) = \sin\theta\left(16\cos^4(\theta) - 12\cos^2(\theta) + 1\right).$$

**15.** Use the formula:

$$1 + z + z^2 + z^3 + z^4 + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$$

to prove Lagrange's identity:

$$\sum_{n=0}^{N}\cos(n\theta) = \frac{1}{2} + \frac{\sin[(N+1/2)\theta]}{2\sin(\theta/2)}.$$