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A Sharp Bound on Eigenvalues of Schrödinger Operators on the Half-line with Complex-valued Potentials

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Abstract. We derive a sharp bound on the location of non-positive eigenvalues of Schrödinger operators on the half-line with complex-valued potentials.

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1. Introduction and main result

In this note we are concerned with estimates for non-positive eigenvalues of one-dimensional Schrödinger operators with complex-valued potentials. We shall provide an example of a bound where the sharp constant *worsens* when a Dirichlet boundary condition is imposed. This is in contrast to the case of real-valued potentials, where the variational principle implies that the absolute value of the non-positive eigenvalues decreases.

In order to describe our result, we first assume that V is real-valued. It is a well-known fact (attributed to L. Spruch in [K]) that any negative eigenvalue λ of the Schrödinger operator $-\partial^2 - V$ in $L^2(\mathbb{R})$ satisfies

$$|\lambda|^{1/2} \leq \frac{1}{2} \int_{-\infty}^{\infty} |V(x)|\, \mathrm{d}x\,. \tag{1.1}$$

The constant $\frac{1}{2}$ in this inequality is sharp and attained if $V(x) = c\delta(x-b)$ for any $c > 0$ and $b \in \mathbb{R}$. (It follows from the Sobolev embedding theorem that the operator $-\partial^2 - V$ can be defined in the quadratic form sense as long as V is a finite Borel measure on $\mathbb{R}$. In this case the right side of (1.1) denotes the total variation of the measure.) From (1.1) and the variational principle for self-adjoint operators

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we immediately infer that any negative eigenvalue of the operator $-\partial^2 - V$ in $L^2(0,\infty)$ with Dirichlet boundary conditions satisfies

$$|\lambda|^{1/2} \le \frac{1}{2} \int_0^\infty |V(x)|\,\mathrm{d}x\,. \tag{1.2}$$

The constant $\frac{1}{2}$ in this inequality is still sharp but no longer attained.

Motivated by concrete physical examples and problems in computational mathematics, an increasing interest in eigenvalue estimates for *complex-valued* potentials has developed in recent years. A beautiful observation of [AAD] is that (1.1) remains valid for all eigenvalues in $\mathbb{C} \setminus [0,\infty)$ even if V is complex-valued. The same is not true for (1.2) ! Indeed, our main result is

Theorem 1.1. *For $a \in \mathbb{R}$ let*

$$g(a) := \sup_{y \ge 0} \left| \mathrm{e}^{iay} - \mathrm{e}^{-y} \right| . \tag{1.3}$$

Any eigenvalue $\lambda = |\lambda| e^{i\theta} \in \mathbb{C} \setminus [0,\infty)$ of the operator $-\partial^2 - V$ in $L^2(0,\infty)$ with Dirichlet boundary conditions satisfies

$$|\lambda|^{1/2} \le \frac{1}{2}\, g(\cot(\theta/2)) \int_0^\infty |V(x)|\,\mathrm{d}x\,. \tag{1.4}$$

This bound is sharp in the following sense: For any given $m > 0$ and $\theta \in (0, 2\pi)$ there are $c \in \mathbb{C}$ and $b > 0$ such that for $V(x) = c\delta(x-b)$ one has $|c| = \int |V(x)|\,\mathrm{d}x = m$ and the unique eigenvalue of $-\partial^2 - V$ is given by $(m^2/4)\, g(\cot(\theta/2))^2 e^{i\theta}$, that is, equality is attained in (1.4).

Remark 1.2. Our bound does not apply to positive eigenvalues. In the case of real-valued potential it is known that there are no positive eigenvalues if $V \in L^1(\mathbb{R})$.

We note that $1 < g(a) < 2$ for $a > 0$. The following lemma discusses the function g in more detail.

Lemma 1.3. *For $a \ge 0$, the function $g(a)$ is monotone increasing, with $g(0) = 1$ and $\lim_{a\to\infty} g(a) = 2$. Moreover,*

$$g(a) = 1 + O(\,\mathrm{e}^{-\pi/(3a)}\,) \tag{1.5}$$

for small a, and

$$g(a) = 2 - \frac{\pi}{a} + O(a^{-2}) \tag{1.6}$$

as $a \to \infty$.

In Figure 1 we plot the curve $\{|z| = g(\cot(\theta/2))^2\}$. It follows from (1.6) that this curve hits the positive real axis at the point 4 with slope $2/\pi$. Close to the point -1 the curve coincides with a semi-circle up to exponentially small terms, as (1.5) shows.

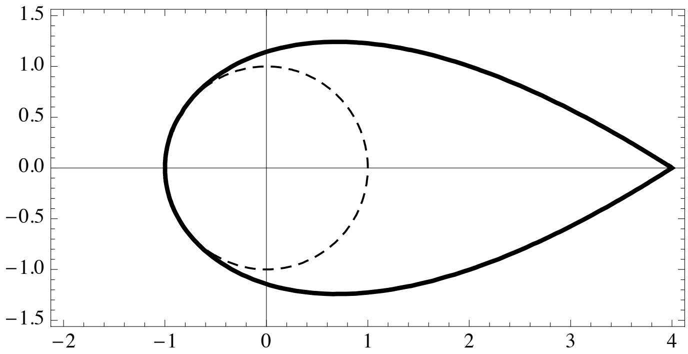

Figure 1. The maximal value of $4|\lambda|$ on the half-line with $\int_0^\infty |V(x)|\,\mathrm{d}x = 1$. The dashed line is the corresponding bound on the whole line.

Using that $\sup_a g(a) = 2$ we find

Corollary 1.4. *Any eigenvalue* $\lambda \in \mathbb{C} \setminus [0,\infty)$ *of the operator* $-\partial^2 - V$ *in* $L^2(0,\infty)$ *with Dirichlet boundary conditions satisfies*

$$|\lambda|^{1/2} \le \int_0^\infty |V(x)|\,\mathrm{d}x\,. \tag{1.7}$$

The bound is not true in general if the right side is multiplied by a constant < 1.

Inequality (1.7) follows also from inequality (1.1) for complex-valued potentials. Indeed, the odd extension of an eigenfunction of the Dirichlet operator is an eigenfunction of the whole-line operator with the potential $V(|x|)$ with the same eigenvalue. The remarkable fact is that the inequality is sharp in the complex-valued case, as shown in Theorem 1.1.

By the same argument (1.7) is also valid if *Neumann* instead of Dirichlet boundary conditions are imposed. In this case equality holds for any $V(x) = c\delta(x)$ with $\operatorname{Re} c > 0$. In particular, in the Neumann case (1.7) is sharp for any fixed argument $0 < \theta < 2\pi$ of the eigenvalue λ. The analogue for mixed boundary conditions is

Proposition 1.5. *Let* $\sigma \ge 0$. *Any eigenvalue* $\lambda \in \mathbb{C} \setminus [0,\infty)$ *of the operator* $-\partial^2 - V$ *in* $L^2(0,\infty)$ *with boundary conditions* $\psi'(0) = \sigma\psi(0)$ *satisfies*

$$|\lambda|^{1/2} \le \int_0^\infty |V(x)|\,\mathrm{d}x\,. \tag{1.8}$$

The bound is sharp for any $\sigma \ge 0$ *and any fixed argument* $0 < \theta < 2\pi$ *of the eigenvalue* λ.

Note that if $\sigma < 0$ a bound of the form (1.8) can not hold since there exists a non-positive eigenvalue even in the case $V = 0$.

Remark 1.6. In the *self-adjoint* case inequality (1.1) for whole-line operators is accompanied by bounds

$$|\lambda|^\gamma \le \frac{\Gamma(\gamma+1)}{\sqrt{\pi}\,\Gamma(\gamma+3/2)} \left(\frac{\gamma-1/2}{\gamma+1/2}\right)^{\gamma-1/2} \int_{-\infty}^{\infty} |V(x)|^{\gamma+1/2}\,\mathrm{d}x \tag{1.9}$$

for $\gamma > 1/2$; see [K, LT]. In contrast, in the *non-selfadjoint* case it seems to be unknown whether the condition $V \in L^{\gamma+1/2}(\mathbb{R})$ for some $1/2 < \gamma < \infty$ implies that all eigenvalues in $\mathbb{C} \setminus [0,\infty)$ lie inside a finite disc; see [DN, FLLS, LS, S] for partial results in this direction. We would like to remark here that even if a bound of the form (1.9) were true in the non-selfadjoint case with $1/2 < \gamma < \infty$, then (in contrast to (1.1) for $\gamma = 1/2$) the constant would have to be strictly larger than in the self-adjoint case. To see this, consider $V(x) = \frac{\alpha(\alpha+1)}{\cosh^2 x}$ with $\operatorname{Re}\alpha > 0$. Then $\lambda = -\alpha^2$ is an eigenvalue (with eigenfunction $(\cosh x)^{-\alpha}$) and the supremum

$$\sup_{\operatorname{Re}\alpha\ge 0} \frac{|\lambda|^\gamma}{\int_{-\infty}^{\infty} |V(x)|^{\gamma+1/2}\,\mathrm{d}x} = \left(\int_{-\infty}^{\infty} \frac{dx}{\cosh^2 x}\right)^{-1} \sup_{\operatorname{Re}\alpha\ge 0} \frac{|\alpha|^{\gamma-1/2}}{|\alpha+1|^{\gamma+1/2}}$$

is clearly attained for purely imaginary values of α.

2. Proofs

Proof of Theorem 1.1. Assume that $-\partial^2\psi(x) - V(x)\psi(x) = -\mu\psi(x)$ with $\psi(0) = 0$, $\psi \not\equiv 0$ and $\mu = -\lambda \in \mathbb{C} \setminus (-\infty, 0]$. Then the Birman-Schwinger operator

$$V^{1/2} \frac{1}{-\partial^2+\mu} |V|^{1/2}, \qquad V^{1/2} := (\operatorname{sgn} V)|V|^{1/2},$$

has an eigenvalue 1, and hence its operator norm is greater or equal to 1.

The integral kernel of this operator equals

$$V(x)^{1/2} \frac{\mathrm{e}^{-\sqrt{\mu}|x-y|} - \mathrm{e}^{-\sqrt{\mu}(x+y)}}{2\sqrt{\mu}} |V(y)|^{1/2},$$

and hence

$$\left|\left(\psi, V^{1/2}\frac{1}{-\partial^2+\mu}|V|^{1/2}\varphi\right)\right| \le \frac{\|V\|_1}{2\sqrt{|\mu|}}\|\psi\|_2\|\varphi\|_2 \sup_{x,y\ge 0}\left|\mathrm{e}^{-\sqrt{\mu}|x-y|} - \mathrm{e}^{-\sqrt{\mu}(x+y)}\right|.$$

Without loss of generality, we can take the supremum over the smaller set $x \ge y \ge 0$. Then

$$\sup_{x\ge y\ge 0}\left|\mathrm{e}^{-\sqrt{\mu}(x-y)} - \mathrm{e}^{-\sqrt{\mu}(x+y)}\right| = \sup_{x\ge y\ge 0} \mathrm{e}^{-x\operatorname{Re}\sqrt{\mu}}\left|\mathrm{e}^{\sqrt{\mu}y} - \mathrm{e}^{-\sqrt{\mu}y}\right|.$$

Since $\operatorname{Re}\sqrt{\mu} > 0$, the supremum over x is achieved at $x = y$, and hence

$$\sup_{x,y\ge 0}\left|\mathrm{e}^{-\sqrt{\mu}(x-y)} - \mathrm{e}^{-\sqrt{\mu}(x+y)}\right| = \sup_{y\ge 0}\left|1 - \mathrm{e}^{-2\sqrt{\mu}y}\right|.$$

If we write $\mu = -|\mu|\,\mathrm{e}^{i\theta}$ with $0 < \theta < 2\pi$, then

$$\sup_{y\geq 0}\left|1 - \mathrm{e}^{-2\sqrt{\mu}y}\right| = \sup_{y\geq 0}\left|\mathrm{e}^{2i\sqrt{|\mu|}\cos(\theta/2)y} - \mathrm{e}^{-2\sqrt{|\mu|}\sin(\theta/2)y}\right| = g(\cot(\theta/2))$$

with g from (1.3). Hence we have shown that

$$\left\|V^{1/2}\frac{1}{-\partial^2+\mu}|V|^{1/2}\right\| \leq \frac{\|V\|_1}{2\sqrt{|\mu|}}\, g(\cot(\theta/2))\,. \tag{2.1}$$

Since the left side is greater or equal to 1, as remarked above, we obtain (1.4).

For $V(x) = c\delta(x-b)$ the Birman-Schwinger operator reduces to the number $c(1-e^{-2\sqrt{\mu}b})/(2\sqrt{\mu})$ and inequality (2.1) becomes equality provided $\sqrt{\mu}b$ satisfies $|1-e^{-2\sqrt{\mu}b}| = g(\cot(\theta/2))$. For given $m > 0$ and $\theta \in (0, 2\pi)$ this determines b and $|c|$. The phase of c is found from the equation $c(1-e^{-2\sqrt{\mu}b})/(2\sqrt{\mu}) = 1$. □

Proof of Lemma 1.3. By continuity for $a > 0$ there exists an optimizer y_0 such that $g(a) = |\,\mathrm{e}^{iay_0} - \mathrm{e}^{-y_0}\,|$. We claim that y_0 satisfies $\pi/3 < ay_0 \leq \pi$. To see the lower bound, note that $|\,\mathrm{e}^{iay} - \mathrm{e}^{-y}\,| \geq 1$ if and only if $2\cos(ay) \leq \mathrm{e}^{-y}$. In particular, $\cos(ay_0) < 1/2$. For the upper bound, if $2\pi > ay > \pi$ and $2\cos(ay) < \mathrm{e}^{-y}$, replacing ya by $2\pi - ya$ leads to a contradiction. Similarly, if $ya > 2\pi$ it can replaced by $ya - 2\pi$ in order exclude that y is the optimizer.

It is elementary to check that $|\,\mathrm{e}^{iay} - \mathrm{e}^{-y}\,|$ is monotone increasing in a for every fixed y with $0 \leq y \leq \pi/a$. Since we know already that $y_0 \leq \pi/a$, the monotonicity of g follows.

Plugging in $y = \pi/a$, we obtain $g(a) \geq 1 + \mathrm{e}^{-\pi/a} \geq 2 - \pi/a$. For large enough a, it follows from this that y_0 is close to π/a. In particular, $y_0 \geq \pi/(2a)$, and hence $|\,\mathrm{e}^{iay_0} - 1| \geq g(a) \geq 2 - \pi/a$. This implies that $y_0 = \pi/a + O(a^{-2})$, and thus $g(a) = 2 - \pi/a + O(a^{-2})$, as claimed.

For an upper bound for small a, we use the triangle inequality and the bound $ay_0 \geq \pi/3$ to find $g(a) \leq 1 + e^{-y_0} \leq 1 + \mathrm{e}^{-\pi/(3a)}$. □

Proof of Proposition 1.5. We proceed as in the proof of Theorem 1.1. The Birman-Schwinger operator has the kernel

$$V(x)^{1/2}\frac{\mathrm{e}^{-\sqrt{\mu}|x-y|} + \frac{\sqrt{\mu}-\sigma}{\sqrt{\mu}+\sigma}\,\mathrm{e}^{-\sqrt{\mu}(x+y)}}{2\sqrt{\mu}}|V(y)|^{1/2}\,.$$

The assertion follows as above using that

$$\sup_{y\geq 0}\left|1 + \frac{\sqrt{\mu}-\sigma}{\sqrt{\mu}+\sigma}\,\mathrm{e}^{-2\sqrt{\mu}y}\right| \leq 2$$

by the triangle inequality and the fact that $|\sqrt{\mu}-\sigma| \leq |\sqrt{\mu}+\sigma|$. The fact that the bound (1.8) is sharp for given argument $0 < \theta < 2\pi$ of the eigenvalue λ follows by choosing $V(x) = -ci\,\mathrm{e}^{i\theta/2}\,\delta(x)$ for $c > 0$ and letting $c \to \infty$. □

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