A BOUND FOR THE EIGENVALUE COUNTING FUNCTION FOR KREIN–VON NEUMANN AND FRIEDRICHS EXTENSIONS

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Dedicated with great pleasure to Yuri Latushkin on the occasion of his 60th birthday.

ABSTRACT. For an arbitrary open, nonempty, bounded set $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, and sufficiently smooth coefficients a, b, q, we consider the closed, strictly positive, higher-order differential operator $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ defined on $W_0^{2m,2}(\Omega)$, associated with the higher-order differential expression

$$\tau_{2m}(a,b,q) := \bigg(\sum_{j,k=1}^{n} (-i\partial_j - b_j) a_{j,k} (-i\partial_k - b_k) + q \bigg)^m, \quad m \in \mathbb{N},$$

and its Krein–von Neumann extension $A_{K,\Omega,2m}(a,b,q)$ in $L^2(\Omega)$. Denoting by $N(\lambda; A_{K,\Omega,2m}(a,b,q))$, $\lambda > 0$, the eigenvalue counting function corresponding to the strictly positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$, we derive the bound

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant C v_n (2\pi)^{-n} \bigg(1 + \frac{2m}{2m+n} \bigg)^{n/(2m)} \lambda^{n/(2m)}, \quad \lambda > 0,$$

where $C = C(a,b,q,\Omega) > 0$ (with $C(I_n,0,0,\Omega) = |\Omega|$) is connected to the eigenfunction expansion of the self-adjoint operator $\widetilde{A}_{2m}(a,b,q)$ in $L^2(\mathbb{R}^n)$ defined on $W^{2m,2}(\mathbb{R}^n)$, corresponding to $\tau_{2m}(a,b,q)$. Here $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ denotes the (Euclidean) volume of the unit ball in $\mathbb{R}^n$.

Our method of proof relies on variational considerations exploiting the fundamental link between the Krein–von Neumann extension and an underlying abstract buckling problem, and on the distorted Fourier transform defined in terms of the eigenfunction transform of $\widetilde{A}_2(a,b,q)$ in $L^2(\mathbb{R}^n)$.

We also consider the analogous bound for the eigenvalue counting function for the Friedrichs extension $A_{F,\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ of $A_{\Omega,2m}(a,b,q)$.

No assumptions on the boundary $\partial\Omega$ of Ω are made.

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1. Introduction

We briefly recall some background material: Suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in a separable complex Hilbert space $\mathcal{H}$ that satisfies

$$S \geqslant \varepsilon I_{\mathcal{H}} \ \text{ for some } \ \varepsilon > 0. \tag{1.1}$$

Then, according to M. Krein's celebrated 1947 paper [59], among all nonnegative self-adjoint extensions of S, there exist two distinguished ones, S_F, the Friedrichs extension of S and S_K, the Krein–von Neumann extension of S, which are, respectively, the largest and smallest such extension (in the sense of quadratic forms). In particular, a nonnegative self-adjoint operator $\widetilde{S}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies

$$S_K \leqslant \widetilde{S} \leqslant S_F \tag{1.2}$$

(again, in the sense of quadratic forms).

An abstract version of [44, Proposition 1], presented in [6], describing the following intimate connection between the nonzero eigenvalues of S_K, and a suitable abstract buckling problem, can be summarized as follows:

There exists $0 \neq v_\lambda \in \mathrm{dom}(S_K)$ satisfying $S_K v_\lambda = \lambda v_\lambda, \quad \lambda \neq 0,$ (1.3)

if and only if

there exists a $0 \neq u_\lambda \in \mathrm{dom}(S^*S)$ such that $S^*S u_\lambda = \lambda S u_\lambda,$ (1.4)

and the solutions v_λ of (1.3) are in one-to-one correspondence with the solutions u_λ of (1.4) given by the pair of formulas

$$u_\lambda = (S_F)^{-1} S_K v_\lambda, \quad v_\lambda = \lambda^{-1} S u_\lambda. \tag{1.5}$$

As briefly recalled in Section 2, (1.4) represents an abstract buckling problem. The latter has been the key in all attempts to date in proving Weyl-type asymptotics for eigenvalues of S_K when S represents an elliptic partial differential operator in $L^2(\Omega)$. In fact, it is convenient to go one step further and replace the abstract buckling eigenvalue problem (1.4) by the variational formulation,

there exists $u_\lambda \in \mathrm{dom}(S) \backslash \{0\}$ such that

$\mathfrak{a}(w, u_\lambda) = \lambda \, \mathfrak{b}(w, u_\lambda)$ for all $w \in \mathrm{dom}(S),$ (1.6)

where the symmetric forms $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{H}$ are defined by

$$\mathfrak{a}(f, g) := (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(\mathfrak{a}) := \mathrm{dom}(S), \tag{1.7}$$

$$\mathfrak{b}(f, g) := (f, Sg)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(\mathfrak{b}) := \mathrm{dom}(S). \tag{1.8}$$

In our present context, the role of the symmetric operator S will be played by the closed, strictly positive operator in $L^2(\Omega)$,

$$A_{\Omega,2m}(a, b, q)f = \tau_{2m}(a, b, q)f, \quad f \in \mathrm{dom}(A_{\Omega,2m}(a, b, q)) := W_0^{2m,2}(\Omega), \tag{1.9}$$

where the differential expression $\tau_{2m}(a, b, q)$ is of the type,

$$\tau_{2m}(a, b, q) := \left(\sum_{j,k=1}^{n} (-i\partial_j - b_j)a_{j,k}(-i\partial_k - b_k) + q \right)^m, \quad m \in \mathbb{N}, \qquad (1.10)$$

under the assumption that $\emptyset \neq \Omega \subset \mathbb{R}^n$ is open and bounded and under sufficient smoothness hypotheses on the coefficients a, b, q (cf. Hypothesis 3.1 (i)). The Krein–von Neumann and Friedrichs extensions of $A_{\Omega, 2m}$ will then be denoted by $A_{K,\Omega,2m}(a, b, q)$ and $A_{F,\Omega,2m}(a, b, q)$, respectively.

Since $A_{K,\Omega,2m}(a, b, q)$ has purely discrete spectrum in $(0, \infty)$ bounded away from zero by $\varepsilon > 0$, let $\{\lambda_{K,\Omega,j}\}_{j \in \mathbb{N}} \subset (0, \infty)$ be the strictly positive eigenvalues of $A_{K,\Omega,2m}(a, b, q)$ enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda; A_{K,\Omega,2m}(a, b, q)) := \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,\Omega,j} < \lambda\}, \quad \lambda > 0, \qquad (1.11)$$

be the eigenvalue distribution function for $A_{K,\Omega,2m}(a, b, q)$ (which takes into account only strictly positive eigenvalues of $A_{K,\Omega,2m}(a, b, q)$); $N(\,\cdot\,; A_{K,\Omega,2m}(a, b, q))$ is the principal object of this note. Similarly, $N(\lambda; A_{F,\Omega,2m}(a, b, q))$, $\lambda > 0$, denotes the eigenvalue counting function for $A_{F,\Omega,2m}(a, b, q)$.

For convenience of the reader, we recall the basic abstract facts on the Friedrichs extension, S_F and the Krein–von Neumann extension S_K of a strictly positive, closed, symmetric operator S in a complex, separable Hilbert space $\mathcal{H}$ and describe the intimate link between the Krein–von Neumann extension and the underlying abstract buckling problem in Section 2. Section 3 focuses on basic domain and spectral properties of the operators, $\widetilde{A}_{2m}(a, b, q)$, $A_{\Omega,2m}(a, b, q)$, $A_{K,\Omega,2m}(a, b, q)$, and $A_{F,\Omega,2m}(a, b, q)$, $m \in \mathbb{N}$, and their associated quadratic forms, on open, bounded subsets $\Omega \subset \mathbb{R}^n$ (without imposing any constraints on $\partial\Omega$). In our principal Section 4 we derive the bounds

$$N(\lambda; A_{K,\Omega,2m}(a, b, q)) \leqslant \frac{v_n}{(2\pi)^n} \left(1 + \frac{2m}{2m+n} \right)^{n/(2m)} \sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|_{L^2(\Omega)}^2 \, \lambda^{n/(2m)},$$
$$\lambda > 0, \qquad (1.12)$$

and

$$N(\lambda; A_{F,\Omega,2m}(a, b, q)) \leqslant \frac{v_n}{(2\pi)^n} \left(1 + \frac{2m}{n} \right)^{n/(2m)} \sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|_{L^2(\Omega)}^2 \, \lambda^{n/(2m)},$$
$$\lambda > 0, \qquad (1.13)$$

where $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ denotes the (Euclidean) volume of the unit ball in $\mathbb{R}^n$ ($\Gamma(\,\cdot\,)$ being the Gamma function), and $\phi(\,\cdot\,, \,\cdot\,)$ represent the suitably normalized generalized eigenfunctions of $\widetilde{A}_2(a, b, q)$ satisfying

$$\widetilde{A}_2(a, b, q)\phi(\,\cdot\,, \xi) = |\xi|^2 \phi(\,\cdot\,, \xi), \quad \xi \in \mathbb{R}^n, \qquad (1.14)$$

in the distributional sense (cf. Hypothesis 4.1). In particular, whenever the property

$$\sup_{(x,\xi) \in \Omega \times \mathbb{R}^n} |\phi(x, \xi)| < \infty \qquad (1.15)$$

has been established, then

$$\sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|_{L^2(\Omega)}^2 \leqslant |\Omega| \sup_{(x,\xi) \in \Omega \times \mathbb{R}^n} \left(|\phi(x, \xi)|^2 \right), \qquad (1.16)$$

explicitly exhibits the volume dependence on Ω of the right-hand sides of (1.12) and (1.13), respectively (see also Section 5).

Our method of employing the eigenfunction transform (i.e., the distorted Fourier transform) associated with the variable coefficient operator $\widetilde{A}_{2m}(a, b, q)$ (replacing the standard Fourier transform in connection with the constant coefficient case in [38]) to derive the results (1.12) and (1.13) appears to be new under any assumptions on $\partial\Omega$. A comparison of (1.12), (1.13) with the existing literature on eigenvalue counting function bounds will be provided in Remark 4.6.

We remark that the power law behavior $\lambda^{n/(2m)}$ coincides with the one in the known Weyl asymptotic behavior. This in itself is perhaps not surprising as it is *a priori* known that

$$N(\lambda; A_{K,\Omega,2m}(a, b, q)) \leqslant N(\lambda; A_{F,\Omega,2m}(a, b, q)), \quad \lambda > 0, \qquad (1.17)$$

and $N(\lambda; A_{F,\Omega,2m}(a, b, q))$ is known to have the power law behavior $\lambda^{n/(2m)}$ (cf. [61] in the case $a = I_n$, $b = q = 0$, extending the corresponding result in [63] in the case $m = 1$). We emphasize that (1.17) is not in conflict with variational eigenvalue estimates since $N(\lambda; A_{K,\Omega,2m}(a, b, q))$ only counts the strictly positive eigenvalues of $A_{K,\Omega,2m}(a, b, q)$ less than $\lambda > 0$ and hence avoids taking into account the (generally, infinite-dimensional) null space of $A_{K,\Omega,2m}(a, b, q)$. Rather than relying on estimates for $N(\,\cdot\,; A_{F,\Omega,2m}(a, b, q))$ (cf., e.g., [12]–[18], [36], [37], [47], [48], [61], [63], [64], [68], [71], [78], [79], [81], [83], [95], typically for $a = I_n$, $b = 0$), we will use the one-to-one correspondence of nonzero eigenvalues of $A_{K,\Omega,2m}(a, b, q)$ with the eigenvalues of its underlying buckling problem (cf. (1.3)–(1.5)) and estimate the eigenvalue counting function for the latter. Section 5 illustrates the purely absolutely continuous spectrum and eigenfunction assumption we impose on $\widetilde{A}_{2m}(a, b, q)$ in $L^2(\mathbb{R}^n)$. Finally, Appendix A derives a crucial minimization result needed in the derivation of the bound (1.12), it also compares (1.12) with the abstract bound (1.17), given (1.13), and points out that the bound (1.12) is always superior to the abstract one guaranteed by combining (1.13) and (1.17).

In the special case $a = I_n$, $b = q = 0$, the bound (1.12) was derived in [38], while the bound (1.13) is due to [61] in this case.

Since Weyl asymptotics for $N(\,\cdot\,; A_{K,\Omega,2m}(a, b, q))$ and $N(\,\cdot\,; A_{F,\Omega,2m}(a, b, q))$ is not considered in this paper (with exception of Remark 4.7), we just refer to the monographs [62] and [84], and to [69], [70], but note that very detailed bibliographies on this subject appeared in [5] and [7]. At any rate, the best known result on Weyl asymptotics with remainder estimate for $N(\,\cdot\,; A_{K,\Omega,2m}(I_n, 0, q))$ to date for bounded Lipschitz domains appears to be [9] (the case of quasi-convex domains having been discussed earlier in [5]). In contrast to Weyl asymptotics with remainder estimates, the estimates (1.12), (1.13) assume no regularity of $\partial\Omega$ at all.

We conclude this introduction by summarizing the notation used in this paper. Throughout this paper, the symbol $\mathcal{H}$ is reserved to denote a separable complex Hilbert space with $(\,\cdot\,, \,\cdot\,)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\mathrm{dom}(T)$ and $\mathrm{ran}(T)$ denoting the domain and range of T. The closure of a closable operator S is denoted by $\overline{S}$. The kernel (null space) of T is denoted by $\ker(T)$. The spectrum, point spectrum (i.e., the set of eigenvalues), discrete spectrum, essential spectrum, and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_d(\cdot)$, $\sigma_{ess}(\cdot)$, and $\rho(\cdot)$,

respectively. The symbol s-lim abbreviates the limit in the strong (i.e., pointwise) operator topology (we also use this symbol to describe strong limits in $\mathcal{H}$).

The Banach spaces of bounded and compact linear operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_p(\mathcal{H})$, $p \in (0, \infty)$. In addition, $U_1 \dotplus U_2$ denotes the direct sum of the subspaces U_1 and U_2 of a Banach space $\mathcal{X}$. Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$.

The symbol $L^2(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$ open, $n \in \mathbb{N}$, is a shortcut for $L^2(\Omega, d^n x)$, whenever the n-dimensional Lebesgue measure $d^n x$ is understood. For brevity, the identity operator in $L^2(\Omega)$ will typically be denoted by I_Ω. The symbol $\mathcal{D}(\Omega)$ is reserved for the set of test functions $C_0^\infty(\Omega)$ on Ω, equipped with the standard inductive limit topology, and $\mathcal{D}'(\Omega)$ represents its dual space, the set of distributions in Ω. The distributional pairing, compatible with the L^2-scalar product, $(\,\cdot\,, \cdot\,)_{L^2(\Omega)}$, is abbreviated by ${}_{\mathcal{D}'(\Omega)}\langle\,\cdot\,, \cdot\,\rangle_{\mathcal{D}(\Omega)}$. The (Euclidean) volume of Ω is denoted by $|\Omega|$.

The cardinality of a set M is abbreviated by $\#M$.

For each multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ (abbreviating $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) we denote by $|\alpha| := \alpha_1 + \cdots + \alpha_n$ the length of α. In addition, we use the standard notations $\partial_j = (\partial/\partial x_j)$, $1 \leqslant j \leqslant n$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$, and $\Delta = \sum_{j=1}^n \partial_j^2$.

2. Basic Facts on the Krein–von Neumann extension and the Associated Abstract Buckling Problem

In this preparatory section we recall the basic facts on the Krein–von Neumann extension of a strictly positive operator S in a complex, separable Hilbert space $\mathcal{H}$ and its associated abstract buckling problem as discussed in [5, 6]. For an extensive survey of this circle of ideas and an exhaustive list of references as well as pertinent historical comments we refer to [7].

To set the stage, we denote by S a linear, densely defined, symmetric (i.e., $S \subseteq S^*$), and closed operator in $\mathcal{H}$ throughout this section. We recall that S is called *nonnegative* provided $(f, Sf)_{\mathcal{H}} \geqslant 0$ for all $f \in \mathrm{dom}(S)$. The operator S is called *strictly positive*, if for some $\varepsilon > 0$ one has $(f, Sf)_{\mathcal{H}} \geqslant \varepsilon \|f\|_{\mathcal{H}}^2$ for all $f \in \mathrm{dom}(S)$; one then writes $S \geqslant \varepsilon I_{\mathcal{H}}$. Next, we recall that two nonnegative, self-adjoint operators A, B in $\mathcal{H}$ satisfy $A \leqslant B$ (in the sense of forms) if

$$\mathrm{dom}\left(B^{1/2}\right) \subset \mathrm{dom}\left(A^{1/2}\right) \tag{2.1}$$

and

$$\left(A^{1/2}f, A^{1/2}f\right)_{\mathcal{H}} \leqslant \left(B^{1/2}f, B^{1/2}f\right)_{\mathcal{H}}, \quad f \in \mathrm{dom}\left(B^{1/2}\right). \tag{2.2}$$

We also recall ([33, Section I.6], [54, Theorem VI.2.21]) that for A and B both self-adjoint and nonnegative in $\mathcal{H}$ one has

$$0 \leqslant A \leqslant B \text{ if and only if } (B + aI_{\mathcal{H}})^{-1} \leqslant (A + aI_{\mathcal{H}})^{-1} \text{ for all } a > 0. \tag{2.3}$$

Moreover, we note the useful fact that $\ker(A) = \ker(A^{1/2})$.

The following is a fundamental result to be found in M. Krein's celebrated 1947 paper [59] (cf. also Theorems 2 and 5–7 in the English summary on page 492):

Theorem 2.1. *Assume that S is a densely defined, closed, nonnegative operator in $\mathcal{H}$. Then, among all nonnegative self-adjoint extensions of S, there exist two*

distinguished ones, S_K and S_F, which are, respectively, the smallest and largest such extension (in the sense of (2.1)–(2.2)). Furthermore, a nonnegative self-adjoint operator $\widetilde{S}$ is a self-adjoint extension of S if and only if $\widetilde{S}$ satisfies

$$S_K \leqslant \widetilde{S} \leqslant S_F. \tag{2.4}$$

In particular, (2.4) determines S_K and S_F uniquely. In addition, if $S \geqslant \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$, one has $S_F \geqslant \varepsilon I_{\mathcal{H}}$, and

$$\operatorname{dom}(S_F) = \operatorname{dom}(S) \dotplus (S_F)^{-1} \ker(S^*), \tag{2.5}$$

$$\operatorname{dom}(S_K) = \operatorname{dom}(S) \dotplus \ker(S^*), \tag{2.6}$$

$$\operatorname{dom}(S^*) = \operatorname{dom}(S) \dotplus (S_F)^{-1} \ker(S^*) \dotplus \ker(S^*)$$
$$= \operatorname{dom}(S_F) \dotplus \ker(S^*), \tag{2.7}$$

and

$$\ker(S_K) = \ker\left((S_K)^{1/2}\right) = \ker(S^*) = \operatorname{ran}(S)^{\perp}. \tag{2.8}$$

One calls S_K the *Krein–von Neumann extension* of S and S_F the *Friedrichs extension* of S. We also recall that

$$S_F = S^*|_{\operatorname{dom}(S^*) \cap \operatorname{dom}((S_F)^{1/2})}. \tag{2.9}$$

Furthermore, if $S \geqslant \varepsilon I_{\mathcal{H}}$ then (2.6) implies

$$\ker(S_K) = \ker\left((S_K)^{1/2}\right) = \ker(S^*) = \operatorname{ran}(S)^{\perp}. \tag{2.10}$$

For abstract results regarding the parametrization of all nonnegative self-adjoint extensions of a given strictly positive, densely defined, symmetric operator we refer the reader to Krein [59], Višik [92], Birman [11], Grubb [42, 43], subsequent expositions due to Alonso and Simon [2], Faris [33, Sect. 15], and [45, Sect. 13.2], [46], [85, Ch. 13], and Derkach and Malamud [27], Malamud [66], see also [41, Theorem 9.2].

Let us collect a basic assumption which will be imposed in the rest of this section.

Hypothesis 2.2. *Suppose that S is a densely defined, symmetric, closed operator with nonzero deficiency indices in $\mathcal{H}$ that satisfies $S \geqslant \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$.*

For subsequent purposes we note that under Hypothesis 2.2, one has

$$\dim\left(\ker(S^* - z I_{\mathcal{H}})\right) = \dim\left(\ker(S^*)\right), \quad z \in \mathbb{C} \backslash [\varepsilon, \infty). \tag{2.11}$$

We recall that two self-adjoint extensions S_1 and S_2 of S are called *relatively prime* (or *disjoint*) if $\operatorname{dom}(S_1) \cap \operatorname{dom}(S_2) = \operatorname{dom}(S)$. The following result will play a role later on (cf., e.g., [5, Lemma 2.8] for an elementary proof):

Lemma 2.3. *Suppose Hypothesis* 2.2. *Then the Friedrichs extension S_F and the Krein–von Neumann extension S_K of S are relatively prime, that is,*

$$\operatorname{dom}(S_F) \cap \operatorname{dom}(S_K) = \operatorname{dom}(S). \tag{2.12}$$

Next, we consider a self-adjoint operator T in $\mathcal{H}$ which is bounded from below, that is, $T \geqslant \alpha I_{\mathcal{H}}$ for some $\alpha \in \mathbb{R}$. We denote by $\{E_T(\lambda)\}_{\lambda \in \mathbb{R}}$ the family of strongly right-continuous spectral projections of T, and introduce for $-\infty \leqslant a < b$, as usual,

$$E_T\left((a,b)\right) = E_T(b_-) - E_T(a) \quad \text{and} \quad E_T(b_-) = \operatorname*{s-lim}_{\varepsilon \downarrow 0} E_T(b - \varepsilon). \tag{2.13}$$

In addition, we set

$$\mu_{T,j} := \inf\left\{\lambda \in \mathbb{R} \,\big|\, \dim(\operatorname{ran}(E_T((-\infty, \lambda)))) \geqslant j\right\}, \quad j \in \mathbb{N}. \tag{2.14}$$

Then, for fixed $k \in \mathbb{N}$, either:

(i) $\mu_{T,k}$ is the kth eigenvalue of T counting multiplicity below the bottom of the essential spectrum, $\sigma_{ess}(T)$, of T,

or,

(ii) $\mu_{T,k}$ is the bottom of the essential spectrum of T,

$$\mu_{T,k} = \inf \left\{ \lambda \in \mathbb{R} \,\middle|\, \lambda \in \sigma_{ess}(T) \right\}, \tag{2.15}$$

and in that case $\mu_{T,k+\ell} = \mu_{T,k}$, $\ell \in \mathbb{N}$, and there are at most $k-1$ eigenvalues (counting multiplicity) of T below $\mu_{T,k}$.

We now record a basic result of M. Krein [59] with an extension due to Alonso and Simon [2] and some additional results recently derived in [6]. For this purpose we introduce the *reduced Krein–von Neumann operator* $\widehat{S}_K$ in the Hilbert space

$$\widehat{\mathcal{H}} := \left(\ker(S^*) \right)^\perp = \left(\ker(S_K) \right)^\perp \tag{2.16}$$

by

$$\widehat{S}_K := P_{(\ker(S_K))^\perp} S_K\big|_{(\ker(S_K))^\perp}, \quad \mathrm{dom}(\widehat{S}_K) = \mathrm{dom}\, S_K \cap \widehat{\mathcal{H}}, \tag{2.17}$$

where $P_{(\ker(S_K))^\perp}$ denotes the orthogonal projection onto $(\ker(S_K))^\perp$. One then obtains

$$\left(\widehat{S}_K \right)^{-1} = P_{(\ker(S_K))^\perp} (S_F)^{-1}\big|_{(\ker(S_K))^\perp}, \tag{2.18}$$

a relation due to Krein [59, Theorem 26] (see also [66, Corollary 5]).

Theorem 2.4. *Suppose Hypothesis 2.2. Then*

$$\varepsilon \leqslant \mu_{S_F,j} \leqslant \mu_{\widehat{S}_K,j}, \quad j \in \mathbb{N}. \tag{2.19}$$

In particular, if the Friedrichs extension S_F of S has purely discrete spectrum, then, except possibly for $\lambda = 0$, the Krein–von Neumann extension S_K of S also has purely discrete spectrum in $(0,\infty)$, that is,

$$\sigma_{ess}(S_F) = \emptyset \ \text{ implies } \ \sigma_{ess}(S_K) \subseteq \{0\}. \tag{2.20}$$

In addition, if $p \in (0,\infty]$, then $(S_F - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_p(\mathcal{H})$ for some $z_0 \in \mathbb{C}\backslash[\varepsilon,\infty)$ implies

$$(S_K - z I_{\mathcal{H}})^{-1}\big|_{(\ker(S_K))^\perp} \in \mathcal{B}_p\big(\widehat{\mathcal{H}}\big) \ \text{ for all } z \in \mathbb{C}\backslash[\varepsilon,\infty). \tag{2.21}$$

In fact, the $\ell^p(\mathbb{N})$-based trace ideal $\mathcal{B}_p(\mathcal{H})$ $\big($resp., $\mathcal{B}_p\big(\widehat{\mathcal{H}}\big)\big)$ of $\mathcal{B}(\mathcal{H})$ $\big($resp., $\mathcal{B}\big(\widehat{\mathcal{H}}\big)\big)$ can be replaced by any two-sided symmetrically normed ideal of $\mathcal{B}(\mathcal{H})$ $\big($resp., $\mathcal{B}\big(\widehat{\mathcal{H}}\big)\big)$.

We note that (2.20) is a classical result of Krein [59]. Apparently, (2.19) in the context of infinite deficiency indices was first proven by Alonso and Simon [2] by a somewhat different method. The implication (2.21) was proved in [6].

Assuming that S_F has purely discrete spectrum, let $\{\lambda_{K,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ be the strictly positive eigenvalues of S_K enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda; S_K) := \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,j} < \lambda\}, \quad \lambda > 0, \tag{2.22}$$

be the eigenvalue distribution function for S_K. Similarly, let $\{\lambda_{F,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ denote the eigenvalues of S_F, again enumerated in nondecreasing order, counting multiplicity, and by

$$N(\lambda; S_F) := \#\{j \in \mathbb{N} \,|\, \lambda_{F,j} < \lambda\}, \quad \lambda > 0, \tag{2.23}$$

the corresponding eigenvalue counting function for S_F. Then inequality (2.19) implies

$$N(\lambda; S_K) \leqslant N(\lambda; S_F), \quad \lambda > 0. \tag{2.24}$$

In particular, any upper estimate for the eigenvalue counting function for the Friedrichs extension S_F, in turn, yields one for the Krein–von Neumann extension S_K (focusing on strictly positive eigenvalues of S_K according to (2.22)). While this is a viable approach to estimate the eigenvalue counting function (2.22) for S_K, we will proceed along a different route in Section 3 and directly exploit the one-to-one corrspondence between strictly positive eigenvalues of S_K and the eigenvalues of its underlying abstract buckling problem to be described next.

To discuss the abstract buckling problem naturally associated with the Krein–von Neumann extension as treated in [6], we start by introducing an abstract version of [44, Proposition 1] (see [6] for a proof):

Lemma 2.5. *Assume Hypothesis 2.2 and let $\lambda \in \mathbb{C}\backslash\{0\}$. Then there exists some $f \in \mathrm{dom}(S_K)\backslash\{0\}$ with*

$$S_K f = \lambda f, \tag{2.25}$$

if and only if there exists $w \in \mathrm{dom}(S^*S)\backslash\{0\}$ such that

$$S^*S w = \lambda S w. \tag{2.26}$$

In fact, the solutions f of (2.25) are in one-to-one correspondence with the solutions w of (2.26) as evidenced by the formulas

$$w = (S_F)^{-1} S_K f, \tag{2.27}$$

$$f = \lambda^{-1} S w. \tag{2.28}$$

Of course, since $S_K \geqslant 0$ is self-adjoint, any $\lambda \in \mathbb{C}\backslash\{0\}$ in (2.25) and (2.26) necessarily satisfies $\lambda \in (0, \infty)$.

It is the linear pencil eigenvalue problem $S^*S w = \lambda S w$ in (2.26) that we call the *abstract buckling problem* associated with the Krein–von Neumann extension S_K of S.

Next, we turn to a variational formulation of the correspondence between the inverse of the reduced Krein–von Neumann extension $\widehat{S}_K$ and the abstract buckling problem in terms of appropriate sesquilinear forms by following [56]–[58] in the elliptic PDE context. This will then lead to an even stronger connection between the Krein–von Neumann extension S_K of S and the associated abstract buckling eigenvalue problem (2.26), culminating in the unitary equivalence result in Theorem 2.6 below.

Given the operator S, we introduce the following symmetric forms in $\mathcal{H}$,

$$\mathfrak{a}(f, g) := (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(\mathfrak{a}) := \mathrm{dom}(S), \tag{2.29}$$

$$\mathfrak{b}(f, g) := (f, Sg)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(\mathfrak{b}) := \mathrm{dom}(S). \tag{2.30}$$

Then S being densely defined and closed implies that the sesquilinear form $\mathfrak{a}$ shares these properties, while $S \geqslant \varepsilon I_{\mathcal{H}}$ from Hypothesis 2.2 implies that $\mathfrak{a}$ is bounded from below, that is,

$$\mathfrak{a}(f, f) \geqslant \varepsilon^2 \|f\|_{\mathcal{H}}^2, \quad f \in \mathrm{dom}(S). \tag{2.31}$$

(The inequality (2.31) follows based on the assumption $S \geqslant \varepsilon I_{\mathcal{H}}$ by estimating $(Sf, Sf)_{\mathcal{H}} = \big([(S - \varepsilon I_{\mathcal{H}}) + \varepsilon I_{\mathcal{H}}]f, [(S - \varepsilon I_{\mathcal{H}}) + \varepsilon I_{\mathcal{H}}]f\big)_{\mathcal{H}}$ from below.)

Thus, one can introduce the Hilbert space

$$\mathcal{W} := \big(\operatorname{dom}(S), (\,\cdot\,,\,\cdot\,)_{\mathcal{W}}\big), \tag{2.32}$$

with associated scalar product

$$(f, g)_{\mathcal{W}} := \mathfrak{a}(f, g) = (Sf, Sg)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(S). \tag{2.33}$$

In addition, we note that $\iota_{\mathcal{W}} : \mathcal{W} \hookrightarrow \mathcal{H}$, the embedding operator of $\mathcal{W}$ into $\mathcal{H}$, is continuous due to $S \geqslant \varepsilon I_{\mathcal{H}}$. Hence, precise notation would be using

$$(w_1, w_2)_{\mathcal{W}} = \mathfrak{a}(\iota_{\mathcal{W}} w_1, \iota_{\mathcal{W}} w_2) = (S\iota_{\mathcal{W}} w_1, S\iota_{\mathcal{W}} w_2)_{\mathcal{H}}, \quad w_1, w_2 \in \mathcal{W}, \tag{2.34}$$

but in the interest of simplicity of notation we will omit the embedding operator $\iota_{\mathcal{W}}$ in the following.

With the sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$ and the Hilbert space $\mathcal{W}$ as above, given $w_2 \in \mathcal{W}$, the map $\mathcal{W} \ni w_1 \mapsto (w_1, Sw_2)_{\mathcal{H}} \in \mathbb{C}$ is continuous. This allows us to define the operator Tw_2 as the unique element in $\mathcal{W}$ such that

$$(w_1, Tw_2)_{\mathcal{W}} = (w_1, Sw_2)_{\mathcal{H}} \text{ for all } w_1 \in \mathcal{W}. \tag{2.35}$$

This implies

$$\mathfrak{a}(w_1, Tw_2) = (w_1, Tw_2)_{\mathcal{W}} = (w_1, Sw_2)_{\mathcal{H}} = \mathfrak{b}(w_1, w_2) \tag{2.36}$$

for all $w_1, w_2 \in \mathcal{W}$. In addition, the operator T satisfies

$$0 \leqslant T = T^* \in \mathcal{B}(\mathcal{W}) \quad \text{and} \quad \|T\|_{\mathcal{B}(\mathcal{W})} \leqslant \varepsilon^{-1}. \tag{2.37}$$

We will call T the *abstract buckling problem operator* associated with the Krein–von Neumann extension S_K of S.

Next, recalling the notation $\widehat{\mathcal{H}} = \big(\ker(S^*)\big)^{\perp}$ (cf. (2.16)), we introduce the operator

$$\widehat{S} : \mathcal{W} \to \widehat{\mathcal{H}}, \quad w \mapsto Sw. \tag{2.38}$$

Clearly, $\operatorname{ran}\big(\widehat{S}\big) = \operatorname{ran}(S)$ and since $S \geqslant \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ and S is closed in $\mathcal{H}$, $\operatorname{ran}(S)$ is also closed, and hence coincides with $\big(\ker(S^*)\big)^{\perp}$. This yields

$$\operatorname{ran}\big(\widehat{S}\big) = \operatorname{ran}(S) = \widehat{\mathcal{H}}. \tag{2.39}$$

In fact, it follows that $\widehat{S} \in \mathcal{B}(\mathcal{W}, \widehat{\mathcal{H}})$ maps $\mathcal{W}$ unitarily onto $\widehat{\mathcal{H}}$ (cf. [6]).

Continuing, we briefly recall the polar decomposition of S,

$$S = U_S |S|, \tag{2.40}$$

where, with $\varepsilon > 0$ as in Hypothesis 2.2,

$$|S| = (S^*S)^{1/2} \geqslant \varepsilon I_{\mathcal{H}} \text{ and } U_S \in \mathcal{B}\big(\mathcal{H}, \widehat{\mathcal{H}}\big) \text{ unitary}. \tag{2.41}$$

Then the principal unitary equivalence result proved in [6] reads as follows:

Theorem 2.6. *Assume Hypothesis* 2.2. *Then the inverse of the reduced Krein–von Neumann extension $\widehat{S}_K$ in $\widehat{\mathcal{H}}$ and the abstract buckling problem operator T in $\mathcal{W}$ are unitarily equivalent. Specifically,*

$$\big(\widehat{S}_K\big)^{-1} = \widehat{S} T \big(\widehat{S}\big)^{-1}. \tag{2.42}$$

In particular, the nonzero eigenvalues of S_K are reciprocals of the eigenvalues of T. Moreover, one has

$$\big(\widehat{S}_K\big)^{-1} = U_S \big[|S|^{-1} S |S|^{-1}\big] (U_S)^{-1}, \tag{2.43}$$

where $U_S \in \mathcal{B}(\mathcal{H}, \widehat{\mathcal{H}})$ is the unitary operator in the polar decomposition (2.40) of S and the operator $|S|^{-1}S|S|^{-1} \in \mathcal{B}(\mathcal{H})$ is self-adjoint and strictly positive in $\mathcal{H}$.

We emphasize that the unitary equivalence in (2.42) is independent of any spectral assumptions on S_K (such as the spectrum of S_K consists of eigenvalues only) and applies to the restrictions of S_K to its pure point, absolutely continuous, and singularly continuous spectral subspaces, respectively.

Equation (2.43) is motivated by rewriting the abstract linear pencil buckling eigenvalue problem (2.26), $S^*Sw = \lambda Sw$, $\lambda \in \mathbb{C}\backslash\{0\}$, in the form

$$|S|^{-1}Sw = (S^*S)^{-1/2}Sw = \lambda^{-1}(S^*S)^{1/2}w = \lambda^{-1}|S|w \qquad (2.44)$$

and hence in the form of a standard eigenvalue problem

$$|S|^{-1}S|S|^{-1}v = \lambda^{-1}v, \quad \lambda \in \mathbb{C}\backslash\{0\}, \quad v := |S|w. \qquad (2.45)$$

Again, self-adjointness and strict positivity of $|S|^{-1}S|S|^{-1}$ imply $\lambda \in (0, \infty)$.

We continue this section with an elementary result (recently noted in [38]) that relates the nonzero eigenvalues of S_K directly with the sesquilinear forms $\mathfrak{a}$ and $\mathfrak{b}$:

Lemma 2.7. *Assume Hypothesis* 2.2 *and introduce*

$$\sigma_p(\mathfrak{a}, \mathfrak{b}) := \big\{ \lambda \in \mathbb{C} \,\big|\, \text{there exists } g_\lambda \in \mathrm{dom}(S)\backslash\{0\}$$
$$\text{such that } \mathfrak{a}(f, g_\lambda) = \lambda\,\mathfrak{b}(f, g_\lambda), \quad f \in \mathrm{dom}(S) \big\}. \qquad (2.46)$$

Then

$$\sigma_p(\mathfrak{a}, \mathfrak{b}) = \sigma_p(S_K)\backslash\{0\} \qquad (2.47)$$

(counting multiplicity), in particular, $\sigma_p(\mathfrak{a}, \mathfrak{b}) \subset (0, \infty)$, *and* $g_\lambda \in \mathrm{dom}(S)\backslash\{0\}$ *in* (2.46) *actually satisfies*

$$g_\lambda \in \mathrm{dom}(S^*S), \quad S^*Sg_\lambda = \lambda Sg_\lambda. \qquad (2.48)$$

In addition,

$$\lambda \in \sigma_p(\mathfrak{a}, \mathfrak{b}) \text{ if and only if } \lambda^{-1} \in \sigma_p(T) \qquad (2.49)$$

(counting multiplicity). Finally,

$$T \in \mathcal{B}_\infty(\mathcal{W}) \iff (\widehat{S}_K)^{-1} \in \mathcal{B}_\infty(\widehat{\mathcal{H}}) \iff \sigma_{ess}(S_K) \subseteq \{0\}, \qquad (2.50)$$

and hence,

$$\sigma_p(\mathfrak{a}, \mathfrak{b}) = \sigma(S_K)\backslash\{0\} = \sigma_d(S_K)\backslash\{0\} \qquad (2.51)$$

if (2.50) *holds. In particular, if one of* S_F *or* $|S|$ *has purely discrete spectrum (i.e.,* $\sigma_{ess}(S_F) = \emptyset$ *or* $\sigma_{ess}(|S|) = \emptyset$), *then* (2.50) *and* (2.51) *hold.*

One notices that $f \in \mathrm{dom}(S)$ in the definition (2.46) of $\sigma_p(\mathfrak{a}, \mathfrak{b})$ can be replaced by $f \in C(S)$ for any (operator) core $C(S)$ for S (equivalently, by any form core for the form $\mathfrak{a}$).

We conclude this section with three auxiliary facts to be used in the proof of Theorem 4.3 and start by recalling an elementary result noted in [38]:

Lemma 2.8. *Suppose that* S *is a densely defined, symmetric, closed operator in* $\mathcal{H}$. *Then* $|S|$ *and hence* S *is infinitesimally bounded with respect to* S^*S, *more precisely, one has*

$$\text{for all } \varepsilon > 0, \ \|Sf\|_{\mathcal{B}(\mathcal{H})} = \||S|f\|_{\mathcal{B}(\mathcal{H})} \leqslant \varepsilon\|S^*Sf\|_{\mathcal{H}}^2 + (4\varepsilon)^{-1}\|f\|_{\mathcal{H}}^2,$$
$$f \in \mathrm{dom}(S^*S). \qquad (2.52)$$

In addition, S is relatively compact with respect to S^*S if $|S|$, or equivalently, S^*S, has compact resolvent. In particular,

$$\sigma_{ess}(S^*S - \lambda S) = \sigma_{ess}(S^*S), \quad \lambda \in \mathbb{R}. \tag{2.53}$$

Given a lower-semibounded, self-adjoint operator $T \geqslant c_T I_{\mathcal{H}}$ in $\mathcal{H}$, we denote by q_T its uniquely associated form, that is,

$$\mathfrak{q}_T(f, g) = \left(|T|^{1/2}f, \operatorname{sgn}(T)|T|^{1/2}g\right)_{\mathcal{H}}, \quad f, g \in \operatorname{dom}(\mathfrak{q}) = \operatorname{dom}\left(|T|^{1/2}\right), \tag{2.54}$$

and by $\{E_T(\lambda)\}_{\lambda \in \mathbb{R}}$ the family of spectral projections of T. We recall the following well-known variational characterization of dimensions of spectral projections $E_T([c_T, \mu))$, $\mu > c_T$.

Lemma 2.9. *Assume that $c_T I_{\mathcal{H}} \leqslant T$ is self-adjoint in $\mathcal{H}$ and $\mu > c_T$. Suppose that $\mathcal{F} \subset \operatorname{dom}\left(|T|^{1/2}\right)$ is a linear subspace such that*

$$\mathfrak{q}_T(f, f) < \mu\|f\|_{\mathcal{H}}^2, \quad f \in \mathcal{F}\backslash\{0\}. \tag{2.55}$$

Then,

$$\dim\left(\operatorname{ran}(E_T([c_T, \mu)))\right) = \sup_{\mathcal{F} \subset \operatorname{dom}(|T|^{1/2})} (\dim(\mathcal{F})). \tag{2.56}$$

We add the following elementary observation: Let $c \in \mathbb{R}$ and $B \geqslant cI_{\mathcal{H}}$ be a self-adjoint operator in $\mathcal{H}$, and introduce the sesquilinear form b in $\mathcal{H}$ associated with B via

$$\begin{aligned} b(u, v) &= \left((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v\right)_{\mathcal{H}} + c(u, v)_{\mathcal{H}}, \\ u, v &\in \operatorname{dom}(b) = \operatorname{dom}\left(|B|^{1/2}\right). \end{aligned} \tag{2.57}$$

Given B and b, one introduces the Hilbert space $\mathcal{H}_b \subseteq \mathcal{H}$ by

$$\begin{aligned} \mathcal{H}_b &= \left(\operatorname{dom}\left(|B|^{1/2}\right), (\cdot, \cdot)_{\mathcal{H}_b}\right), \\ (u, v)_{\mathcal{H}_b} &= b(u, v) + (1 - c)(u, v)_{\mathcal{H}} \\ &= \left((B - cI_{\mathcal{H}})^{1/2}u, (B - cI_{\mathcal{H}})^{1/2}v\right)_{\mathcal{H}} + (u, v)_{\mathcal{H}} \\ &= \left((B + (1 - c)I_{\mathcal{H}})^{1/2}u, (B + (1 - c)I_{\mathcal{H}})^{1/2}v\right)_{\mathcal{H}}. \end{aligned} \tag{2.58}$$

One observes that

$$(B + (1 - c)I_{\mathcal{H}})^{1/2} \colon \mathcal{H}_b \to \mathcal{H} \text{ is unitary.} \tag{2.59}$$

Finally, we recall the following fact (cf., e.g., [40]).

Lemma 2.10. *Let $\mathcal{H}$, B, b, and $\mathcal{H}_b$ be as in (2.57)–(2.59). Then B has purely discrete spectrum, that is, $\sigma_{ess}(B) = \emptyset$, if and only if $\mathcal{H}_b$ embeds compactly into $\mathcal{H}$.*

3. Preliminaries on a Class of Partial Differential Operators

In this section we set the stage for our principal results in the next Section 4 and introduce the class of even-order partial differential operators $\widetilde{A}_{2m}(a, b, q)$ in $L^2(\mathbb{R}^n)$ as well as $A_{\Omega, 2m}(a, b, q)$ in $L^2(\Omega)$ (see (3.6) for the underlying differential expressions), with $\emptyset \neq \Omega \subset \mathbb{R}^n$ open and bounded (but no other conditions imposed on Ω). In particular, we provide a detailed study of their domains and quadratic form domains, including spectral properties such as strict boundedness from below for the Friedrichs extension $A_{F, \Omega, 2m}(a, b, q)$ of $A_{\Omega, 2m}(a, b, q)$ in $L^2(\Omega)$, employing a diamagnetic inequality.

Hypothesis 3.1. (i) *Let* $m \in \mathbb{N}$. *Assume that*

$$b = (b_1, b_2, \ldots, b_n) \in \left[W^{(2m-1),\infty}(\mathbb{R}^n) \right]^n, \quad b_j \ \text{real-valued}, \ 1 \leqslant j \leqslant n, \qquad (3.1)$$

$$0 \leqslant q \in W^{(2m-2),\infty}(\mathbb{R}^n). \qquad (3.2)$$

Suppose that $a := \{a_{j,k}\}_{1 \leqslant j,k \leqslant n}$ *is a real symmetric matrix satisfying*

$$a_{j,k} \in C^{(2m-1)}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad 1 \leqslant j, k \leqslant n, \qquad (3.3)$$

and that there exists $\varepsilon_a > 0$, *such that*

$$\sum_{j,k=1}^n a_{j,k}(x) y_j y_k \geqslant \varepsilon_a |y|^2, \quad x \in \mathbb{R}^n, \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n. \qquad (3.4)$$

(ii) *Let* $\emptyset \neq \Omega \subset \mathbb{R}^n$ *be open and bounded. In addition, assume that the* $n \times n$ *matrix-valued function* a *equals the identity* I_n *outside a ball* $B_n(0; R_0)$ *containing* $\overline{\Omega}$, *that is, there exists* $R_0 > 0$ *such that*

$$a(x) = I_n, \quad |x| \geqslant R_0, \quad \overline{\Omega} \subset B_n(0; R_0). \qquad (3.5)$$

For simplicity we introduced the ball $B_n(0; R_0)$ containing $\overline{\Omega}$ in Hypothesis 3.1 (ii), but for any fixed $\varepsilon > 0$, one can of course replace $B_n(0; R_0)$ by an open ε-neighborhood Ω_ε of $\overline{\Omega}$.

We will consider various closed (and self-adjoint) L^2-realizations of the differential expression

$$\tau_{2m}(a, b, q) := \left(\sum_{j,k=1}^n (-i\partial_j - b_j(x)) a_{j,k}(x) (-i\partial_k - b_k(x)) + q(x) \right)^m, \qquad (3.6)$$
$$m \in \mathbb{N}, \ x \in \mathbb{R}^n.$$

We note that Hypothesis 3.1 (i) was of course chosen with $\tau_{2m}(a, b, q)$ in mind. In some instances we only consider the special case $m = 1$, that is, $\tau_2(a, b, q)$, and then choosing the most general case $m = 1$ in Hypothesis 3.1 (i) will of course be sufficient. We will tacitly assume such a relaxation of hypotheses on the coefficients a, b, q without necessarily dwelling on this explicitly in every such instance.

In the following we find it convenient using auxiliary operators corresponding to the leading and the lower-order terms of the differential expression (3.6). To this end we first introduce the differential expression $\tau_{2m}(a) = \tau_{2m}(a, 0, 0)$,

$$\tau_{2m}(a) := \left(- \sum_{j,k=1}^n \partial_j a_{j,k}(x) \partial_k \right)^m, \quad m \in \mathbb{N}, \ x \in \mathbb{R}^n, \qquad (3.7)$$

and the associated linear operator $\widetilde{T}_{2m}(a)$ in $L^2(\mathbb{R}^n)$ given by

$$\widetilde{T}_{2m}(a)u := \tau_{2m}(a)u, \quad u \in \mathrm{dom}\,(\widetilde{T}_{2m}(a)) := W^{2m,2}(\mathbb{R}^n). \qquad (3.8)$$

Second, we observe that due to boundedness of the coefficients a, b, q (cf. (3.1)) and sufficiently many of their derivatives, one has

$$\tau_{2m}(a, b, q)u = \tau_{2m}(a)u + \sum_{0 \leqslant |\alpha| \leqslant 2m-1} g_\alpha(a, b, q, x) \partial^\alpha u,$$
$$\tau_{2m}(a, b, q)u \in L^2(\mathbb{R}^n), \quad u \in W^{2m,2}(\mathbb{R}^n), \qquad (3.9)$$

for some $g_\alpha(a, b, q, \cdot) \in L^\infty(\mathbb{R}^n)$, $0 \leqslant |\alpha| \leqslant 2m - 1$. The sum of the lower-order terms in (3.9) gives rise to a linear operator $\widetilde{S}_{2m-1}(a, b, q)$ in $L^2(\mathbb{R}^n)$,

$$\widetilde{S}_{2m-1}(a, b, q)u := \sum_{0 \leqslant |\alpha| \leqslant 2m-1} g_\alpha(a, b, q, x)\partial^\alpha u,$$

$$u \in \mathrm{dom}\left(\widetilde{S}_{2m-1}(a, b, q)\right) := W^{2m,2}(\mathbb{R}^n). \tag{3.10}$$

Next, we introduce the operator $\widetilde{A}_{2m}(a, b, q)$ in $L^2(\mathbb{R}^n)$ by

$$\widetilde{A}_{2m}(a, b, q)u := \tau_{2m}(a, b, q)u, \quad u \in \mathrm{dom}\left(\widetilde{A}_{2m}(a, b, q)\right) := W^{2m,2}(\mathbb{R}^n), \tag{3.11}$$

and its restriction $\widetilde{A}_{0,2m}(a, b, q)$ to $C_0^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ via

$$\widetilde{A}_{0,2m}(a, b, q)u := \tau_{2m}(a, b, q)u, \quad u \in \mathrm{dom}\left(\widetilde{A}_{0,2m}(a, b, q)\right) := C_0^\infty(\mathbb{R}^n). \tag{3.12}$$

Making use of standard perturbation results, it is convenient to view the operator $\widetilde{A}_{2m}(a, b, q)$ as perturbation of $\widetilde{T}_{2m}(a)$ by $\widetilde{S}_{2m-1}(a, b, q)$ and state the following auxiliary fact.

Theorem 3.2. *Assume Hypothesis 3.1 (i). Then $\widetilde{A}_{0,2m}(a, b, q)$ is essentially self-adjoint in $L^2(\mathbb{R}^n)$, its closure equals $\widetilde{A}_{2m}(a, b, q)$, and hence,*

$$\widetilde{A}_{2m}(a, b, q) \geqslant 0. \tag{3.13}$$

In addition, the graph norm of $\widetilde{A}_{2m}(a, b, q)$ is equivalent to the norm of the Sobolev space $W^{2m,2}(\mathbb{R}^n)$, that is, there exist constants $0 < c < C$, depending only on a, b, q, m, n, such that

$$c\|u\|_{W^{2m,2}(\mathbb{R}^n)}^2 \leqslant \left\|\widetilde{A}_{2m}(a, b, q)u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2 \leqslant C\|u\|_{W^{2m,2}(\mathbb{R}^n)}^2,$$

$$u \in W^{2m,2}(\mathbb{R}^n). \tag{3.14}$$

Proof. We introduce the minimal operator $\widetilde{T}_{0,2m}(a)$ in $L^2(\mathbb{R}^n)$ by

$$\widetilde{T}_{0,2m}(a)u := \tau_{2m}(a)u, \quad u \in \mathrm{dom}(\widetilde{T}_{0,2m}(a)) := C_0^\infty(\mathbb{R}^n), \tag{3.15}$$

and will show that it is essentially self-adjoint and that $\widetilde{T}_{2m}(a) = \left(\widetilde{T}_{0,2m}(a)\right)^*$; the operator $\widetilde{A}_{2m}(a, b, q)$ will then be considered as an infinitesimally bounded perturbation of $\widetilde{T}_{2m}(a)$.

Let $u \in L^2(\mathbb{R}^n) \cap W_{loc}^{2m,2}(\mathbb{R}^n)$ and $\tau_{2m}(a)u \in L^2(\mathbb{R}^n)$, then for arbitrary $v \in \mathrm{dom}\left(\widetilde{T}_{0,2m}(a)\right) = C_0^\infty(\mathbb{R}^n)$ one has

$$\left(u, \widetilde{T}_{0,2m}(a)v\right)_{L^2(\mathbb{R}^n)} = (u, \tau_{2m}(a)v)_{L^2(\mathbb{R}^n)}$$

$$= {}_{\mathcal{D}'(\mathbb{R}^n)}\langle \tau_{2m}(a)u, v \rangle_{\mathcal{D}(\mathbb{R}^n)} = (\tau_{2m}(a)u, v)_{L^2(\mathbb{R}^n)}, \tag{3.16}$$

hence $u \in \mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*\right)$ and $\left(\widetilde{T}_{0,2m}(a)\right)^* u = \tau_{2m}(a)u$, implying

$$\left\{u \in L^2(\mathbb{R}^n) \,\middle|\, u \in W_{loc}^{2m,2}(\mathbb{R}^n), \, \tau_{2m}(a)u \in L^2(\mathbb{R}^n)\right\} \subseteq \mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*\right). \tag{3.17}$$

Using the interior regularity for elliptic differential operators, one obtains the converse inclusion: Indeed, if $u \in \mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*\right)$, then $u \in L^2(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ and for some $v \in L^2(\mathbb{R}^n)$ one has $\tau_{2m}(a)u = v$, implying $u \in W_{loc}^{2m,2}(\mathbb{R}^n)$ (see, e.g., [73, Theorem 1.3], see also [90]).

Our next objective is to show that $\mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*\right) = W^{2m,2}(\mathbb{R}^n)$. Let $\varphi_{R_0} \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_{R_0}(x) = 1$, $x \in B_n(0; R_0)$, cf. (3.5). Since $u\varphi_{R_0} \in W^{2m,2}(\mathbb{R}^n)$ for

any $u \in \mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*\right)$, in order to prove that $\mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*\right) \subseteq W^{2m,2}(\mathbb{R}^n)$ it suffices to obtain the inclusion $u(1 - \varphi_{R_0}) \in W^{2m,2}(\mathbb{R}^n)$. This, in turn, will be guaranteed once we prove the following fact,

$$\mathrm{dom}\left(\left(\widetilde{T}_{0,2m}(a)\right)^*(1 - \varphi_{R_0})\right) = \mathrm{dom}\left(H_0^m(1 - \varphi_{R_0})\right). \tag{3.18}$$

Here the self-adjoint operator H_0 in $L^2(\mathbb{R}^n)$ is defined by

$$H_0 u = (-\Delta)u, \quad u \in \mathrm{dom}(H_0) = W^{2,2}(\mathbb{R}^n), \tag{3.19}$$

and hence

$$H_0^\alpha u = (-\Delta)^\alpha u, \quad u \in \mathrm{dom}\left(H_0^\alpha\right) = W^{2\alpha,2}(\mathbb{R}^n), \quad \alpha \in (0, \infty). \tag{3.20}$$

For $u \in \mathrm{dom}(H_0^m(1 - \varphi_{R_0}))$, the expression $\left(\widetilde{T}_{0,2m}(a)\right)^*(1 - \varphi_{R_0})u - H_0^m(1 - \varphi_{R_0})u$ does not contain derivatives of u of order higher than $2m - 1$, therefore, for any $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that

$$\left\|\left(\widetilde{T}_{0,2m}(a)\right)^*(1 - \varphi_{R_0})u - H_0^m(1 - \varphi_{R_0})u\right\|_{L^2(\mathbb{R}^n)}^2$$
$$\leqslant \varepsilon\left\|H_0^m(1 - \varphi_{R_0})u\right\|_{L^2(\mathbb{R}^n)}^2 + k(\varepsilon)\|u\|_{L^2(\mathbb{R}^n)}^2, \quad u \in \mathrm{dom}\left(H_0^m(1 - \varphi_{R_0})\right). \tag{3.21}$$

Combining (3.21) and [54, Theorem IV 1.1] one obtains equality of the domains in (3.18), and hence also $\mathrm{dom}((\widetilde{T}_{0,2m}(a))^*) \subseteq W^{2m,2}(\mathbb{R}^n)$. The opposite inclusion is clear from (3.17).

Next we will show that

$$(\widetilde{T}_{0,2m}(a))^* u = \tau_{2m}(a)u, \quad u \in \mathrm{dom}\left((\widetilde{T}_{0,2m}(a))^*\right) = W^{2m,2}(\mathbb{R}^n). \tag{3.22}$$

To this end we fix $v \in \mathrm{dom}\left(\widetilde{T}_{0,2m}(a)\right) = C_0^\infty(\mathbb{R}^n)$. Then using the inclusion $\tau_{2m}(a)u \in L^2(\mathbb{R}^n)$, one obtains

$$\left(u, \widetilde{T}_{0,2m}(a)v\right)_{L^2(\mathbb{R}^n)} = (u, \tau_{2m}(a)v)_{L^2(\mathbb{R}^n)}$$
$$= {}_{\mathcal{D}'(\mathbb{R}^n)}\langle \tau_{2m}(a)u, v \rangle_{\mathcal{D}(\mathbb{R}^n)} = (\tau_{2m}(a)u, v)_{L^2(\mathbb{R}^n)}, \tag{3.23}$$

and hence $\left(\widetilde{T}_{0,2m}(a)\right)^* u = \tau_{2m}(a)u$, implying that $(\widetilde{T}_{0,2m}(a))^*$ is symmetric. Therefore $\widetilde{T}_{0,2m}(a)$ is essentially self-adjoint and thus $\widetilde{T}_{2m}(a) = \left(\widetilde{T}_{0,2m}(a)\right)^*$ is self-adjoint.

The proof thus far showed an important fact: The graph norms of the operators $\widetilde{T}_{2m}(a)$ and H_0^m, both defined on $W^{2m,2}(\mathbb{R}^n)$, are equivalent, that is, there exist constants $0 < c_1 < C_1$, depending only on the coefficients a, b, q, m, n, such that

$$c_1\left[\left\|H_0^m u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2\right] \leqslant \left\|\widetilde{T}_{2m}(a)u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2$$
$$\leqslant C_1\left[\left\|H_0^m u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2\right], \quad u \in W^{2m,2}(\mathbb{R}^n). \tag{3.24}$$

In particular, the graph norm of $\widetilde{T}_{2m}(a)$ is equivalent to the norm of $W^{2m,2}(\mathbb{R}^n)$.

Finally we show that $\left(\widetilde{A}_{0,m}(a,b,q)\right)^*$ is symmetric, actually, self-adjoint, proving that $\widetilde{A}_{0,m}(a,b,q)$ is essentially self-adjoint. To this end, we recall the operator $\widetilde{S}_{2m-1}(a,b,q)$ in (3.10), corresponding to lower-order terms in the differential expression $\tau_{2m}(a,b,q)$. Since $\widetilde{S}_{2m-1}(a,b,q)$ has bounded coefficients and its order is

at most $2m - 1$, it is infinitesimally bounded with respect to the polyharmonic operator H_0^m. Thus, for any $\varepsilon > 0$ there exists $k(\varepsilon) > 0$ such that

$$\left\|\widetilde{S}_{2m-1}(a,b,q)u\right\|_{L^2(\mathbb{R}^n)}^2 \leqslant \varepsilon\|H_0^m u\|_{L^2(\mathbb{R}^n)}^2 + k(\varepsilon)\|u\|_{L^2(\mathbb{R}^n)}^2, \quad u \in W^{2m,2}(\mathbb{R}^n). \tag{3.25}$$

Combining this inequality with the equivalence of the graph norms of $\widetilde{T}_{2m}(a)$ and H_0^m, one concludes that $\widetilde{S}_{2m-1}(a,b,q)$ is infinitesimally bounded with respect to $\widetilde{T}_{2m}(a)$. Hence, $\widetilde{A}_{0,2m}(a,b,q) = \widetilde{T}_{0,2m}(a) + \widetilde{S}_{2m-1}(a,b,q)$ is essentially self-adjoint, and $\mathrm{dom}((A_{0,m}(a,b,q))^*) = \mathrm{dom}\big(\widetilde{T}_{2m}(a)\big) = W^{2m,2}(\mathbb{R}^n)$. The fact (3.14) follows from [28, Proposition 7.2] and the fact that $\widetilde{A}_{2m}(a,b,q)$ and H_0^m have the common domain $W^{2m,2}(\mathbb{R}^n)$ and both are closed (in fact, self-adjoint). $\qquad\square$

Lemma 3.3. *Assume Hypothesis* 3.1 (*i*)*. Then for all* $\alpha \in (0,1]$,

$$\mathrm{dom}\left(\big(\widetilde{A}_{2m}(a,b,q)\big)^\alpha\right) = W^{2m\alpha,2}(\mathbb{R}^n), \tag{3.26}$$

and there exist constants $0 < c < C$ *depending only on* a,b,q,m,n, *such that*

$$c\|u\|_{W^{m,2}(\mathbb{R}^n)}^2 \leqslant \left\|\widetilde{A}_{2m}(a,b,q)^{1/2}u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2 \leqslant C\|u\|_{W^{m,2}(\mathbb{R}^n)}^2, \\ u \in W^{m,2}(\mathbb{R}^n), \tag{3.27}$$

and hence,

$$c\|u\|_{W^{m,2}(\mathbb{R}^n)}^2 \leqslant \big(u, \widetilde{A}_{2m}(a,b,q)u\big)_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}^2 \leqslant C\|u\|_{W^{m,2}(\mathbb{R}^n)}^2, \\ u \in W^{2m,2}(\mathbb{R}^n). \tag{3.28}$$

Proof. We start with a well-known interpolation argument: Let S and T be closed operators in $\mathcal{H}$ satisfying $\mathrm{dom}(S) \supseteq \mathrm{dom}(T)$. Then S is relatively bounded with respect to T (cf., e.g., [28, Proposition III.7.2], [54, Remark IV.1.5]) and hence there exist $a > 0$ and $b > 0$ such that

$$\||S|f\|_\mathcal{H}^2 = \|Sf\|_\mathcal{H}^2 \leqslant a^2\|Tf\|_\mathcal{H}^2 + b^2\|f\|_\mathcal{H}^2 = a^2\||T|f\|_\mathcal{H}^2 + b^2\|f\|_\mathcal{H}^2$$
$$= \left\|\big[a^2|T|^2 + b^2 I_\mathcal{H}\big]^{1/2}f\right\|_\mathcal{H}^2, \quad f \in \mathrm{dom}(T) = \mathrm{dom}(|T|). \tag{3.29}$$

Thus, applying the Loewner–Heinz inequality (cf., e.g., [51], [60, Theorem IV.1.11]), one infers that (see also [39])

$$\mathrm{dom}\left(|S|^\alpha\right) \supseteq \mathrm{dom}\left(\big(a^2|T|^2 + b^2 I_\mathcal{H}\big)^{\alpha/2}\right) = \mathrm{dom}\left(|T|^\alpha\right), \quad \alpha \in (0,1]. \tag{3.30}$$

In particular, if $\mathrm{dom}(S) = \mathrm{dom}(T)$ one concludes that

$$\mathrm{dom}\left(|S|^\alpha\right) = \mathrm{dom}\left(|T|^\alpha\right), \quad \alpha \in (0,1]. \tag{3.31}$$

Identifying S with $\widetilde{A}_{2m}(a,b,q)$ and T with H_0^m, (3.20) and (3.31) prove (3.26).

Employing (3.26) with $\alpha = 1/2$ one infers that

$$\left\|\widetilde{A}_{2m}(a,b,q)^{1/2}u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2 \approx \left\|H_0^{m/2}u\right\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2, \\ u \in W^{m,2}(\mathbb{R}^n). \tag{3.32}$$

Assuming, in addition, that $u \in W^{2m,2}(\mathbb{R}^n)$, (3.32) can be rewritten as

$$\big(u, \widetilde{A}_{2m}(a,b,q)u\big)_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}^2 \approx \big(u, H_0^m u\big)_{L^2(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)}^2, \tag{3.33}$$

and together with the fact that the right-hand side of (3.33) is equivalent to the norm $\|\cdot\|_{W^{m,2}(\mathbb{R}^n)}^2$, one arrives at (3.28). $\qquad\square$

Given Lemma 3.3, the sequilinear form $Q_{\widetilde{A}_{2m}(a,b,q)}$ in $L^2(\mathbb{R}^n)$ associated with $\widetilde{A}_{2m}(a,b,q)$ is given by

$$
\begin{aligned}
Q_{\widetilde{A}_{2m}(a,b,q)}(u,v) &:= \big(\widetilde{A}_{2m}(a,b,q)^{1/2}u, \widetilde{A}_{2m}(a,b,q)^{1/2}v\big)_{L^2(\mathbb{R}^n)}, \\
u,v &\in \mathrm{dom}(Q_{\widetilde{A}_{2m}(a,b,q)}) = \mathrm{dom}\big(\widetilde{A}_{2m}(a,b,q)^{1/2}\big) = W^{m,2}(\mathbb{R}^n),
\end{aligned}
\tag{3.34}
$$

and we also introduce

$$
Q_{H_0^m}(u,v) := \big(H_0^{m/2}u, H_0^{m/2}v\big)_{L^2(\mathbb{R}^n)}, \quad u,v \in \mathrm{dom}(Q_{H_0^m}) = W^{m,2}(\mathbb{R}^n). \tag{3.35}
$$

In addition, we will employ the explicit representation of the form $Q_{\widetilde{A}_{2m}(a,b,q)}$ in terms of $\widetilde{A}_{2m}(a,b,q)$,

$$
Q_{\widetilde{A}_{2m}(a,b,q)}(u,v) = \begin{cases} (\tau_{2\ell}u, \tau_{2\ell}v)_{L^2(\mathbb{R}^n)}, \quad m = 2\ell,\ \ell \in \mathbb{N}, \\ \sum_{j,k=1}^n ((-i\partial_j - b_j)\tau_{2\ell}u, a_{j,k}(-i\partial_k - b_k)\tau_{2\ell}v)_{L^2(\mathbb{R}^n)} \\ + (\tau_{2\ell}u, q\,\tau_{2\ell}v)_{L^2(\mathbb{R}^n)}, \quad m = 2\ell+1,\ \ell \in \mathbb{N}\cup\{0\}, \end{cases}
$$
$$
u,v \in W^{m,2}(\mathbb{R}^n). \tag{3.36}
$$

Here, in obvious notation, $\tau_0 = 1$.

Assuming Hypothesis 3.1 (i), we introduce one of the main objects of our study, the symmetric operator $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ by

$$
A_{\Omega,2m}(a,b,q)f = \tau_{2m}(a,b,q)f, \quad f \in \mathrm{dom}(A_{\Omega,2m}(a,b,q)) = W_0^{2m,2}(\Omega), \tag{3.37}
$$

and note that $\widetilde{A}_{2m}(a,b,q)$ formally represents its extended version in $L^2(\mathbb{R}^n)$. In addition, we introduce the associated minimal operator $A_{min,\Omega,2m}(a,b,q)$ in $L^2(\Omega)$ by

$$
\begin{aligned}
A_{min,\Omega,2m}(a,b,q)f &:= \tau_{2m}(a,b,q)f, \\
f &\in \mathrm{dom}(A_{min,\Omega,2m}(a,b,q)) := C_0^\infty(\Omega).
\end{aligned}
\tag{3.38}
$$

Clearly, $A_{min,\Omega,2m}(a,b,q)$ is symmetric (hence, closable) in $L^2(\Omega)$ (upon elementary integration by parts) and nonnegative,

$$
A_{min,\Omega,2m}(a,b,q) \geqslant 0. \tag{3.39}
$$

Theorem 3.4. *Assume Hypothesis* 3.1 (i). *Then the closure of* $A_{min,\Omega,2m}(a,b,q)$ *in* $L^2(\Omega)$ *is given by* $A_{\Omega,2m}(a,b,q)$,

$$
\overline{A_{min,\Omega,2m}(a,b,q)} = A_{\Omega,2m}(a,b,q). \tag{3.40}
$$

In particular, $A_{\Omega,2m}(a,b,q)$ *is symmetric and nonnegative in* $L^2(\Omega)$,

$$
A_{\Omega,2m}(a,b,q) \geqslant 0. \tag{3.41}
$$

In addition, there exist $0 < c < C$, *depending only on* a,b,q,m,n, *such that*

$$
c\|f\|_{W_0^{2m,2}(\Omega)}^2 \leqslant \|A_{\Omega,2m}(a,b,q)f\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \leqslant C\|f\|_{W_0^{2m,2}(\Omega)}^2, \tag{3.42}
$$
$$
f \in W_0^{2m,2}(\Omega).
$$

Proof. Using (3.14) with $v \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}(v) \subset \Omega$ one concludes that the graph norm of $A_{\Omega,2m}(a,b,q)$ is equivalent to the norm of $\mathring{W}^{2m,2}(\Omega)$ on $C_0^\infty(\Omega)$. Therefore, $\mathrm{dom}\big(\overline{A_{min,\Omega,2m}(a,b,q)}\big) = \mathring{W}^{2m,2}(\Omega)$. In order to prove that $\overline{A_{min,\Omega,2m}(a,b,q)}f = \tau_{2m}(a,b,q)f$, we consider $\{f_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$, $f,g \in L^2(\Omega)$, such that

$$
\lim_{j\to\infty} \|f_j - f\|_{L^2(\Omega)} = 0 \text{ and } \lim_{j\to\infty} \big\|A_{min,\Omega,2m}(a,b,q)f_j - g\big\|_{L^2(\Omega)} = 0. \tag{3.43}
$$

Since $A_{min,\Omega,2m}(a,b,q)$ is symmetric and hence closable in $L^2(\Omega)$, one infers that

$$f \in \text{dom}\left(\overline{A_{min,\Omega,2m}(a,b,q)}\right) = W_0^{2m,2}(\Omega) \text{ and } \overline{A_{min,\Omega,2m}(a,b,q)}f = g. \quad (3.44)$$

Taking arbitrary $\psi \in C_0^\infty(\Omega)$, and recalling our notation for the distributional pairing $_{\mathcal{D}'(\Omega)}\langle \cdot, \cdot \rangle_{\mathcal{D}(\Omega)}$ (compatible with the scalar product $(\cdot, \cdot)_{L^2(\Omega)}$), one concludes that

$$(g,\psi)_{L^2(\Omega)} = {}_{\mathcal{D}'(\Omega)}\langle g, \psi \rangle_{\mathcal{D}(\Omega)} = \lim_{j\to\infty} {}_{\mathcal{D}'(\Omega)}\langle \tau_{2m}(a,b,q)f_j, \psi \rangle_{\mathcal{D}(\Omega)}$$

$$= \lim_{j\to\infty} \int_\Omega \overline{f_j(x)}\left(\tau_{2m}(a,b,q)\psi\right)(x)\,d^nx = \int_\Omega \overline{f(x)}\left(\tau_{2m}(a,b,q)\psi\right)(x)\,d^nx$$

$$= {}_{\mathcal{D}'(\Omega)}\langle \tau_{2m}(a,b,q)f, \psi \rangle_{\mathcal{D}(\Omega)}, \quad (3.45)$$

implying $g = \tau_{2m}(a,b,q)f$ and hence, $\overline{A_{min,\Omega,2m}(a,b,q)}f = A_{\Omega,2m}(a,b,q)f$ implying (3.40). This also completes the proof of (3.42). Finally, being the closure of the symmetric operator $A_{min,\Omega,2m}(a,b,q)$, also $A_{\Omega,2m}(a,b,q)$ is symmetric in $L^2(\Omega)$ (cf., e.g., [96, Theorem 5.4 (b)]). $\square$

Next, still assuming Hypothesis 3.1 (i), we introduce the form $Q_{A_{\Omega,2m}(a,b,q)}$ in $L^2(\Omega)$ generated by $A_{\Omega,2m}(a,b,q)$, via

$$Q_{A_{\Omega,2m}(a,b,q)}(f,g) := (f, A_{\Omega,2m}(a,b,q)g)_{L^2(\mathbb{R}^n)},$$
$$f,g \in \text{dom}(Q_{A_{\Omega,2m}(a,b,q)}) := W_0^{2m,2}(\Omega). \quad (3.46)$$

Lemma 3.5. *Assume Hypothesis 3.1 (i). Then the form $Q_{A_{\Omega,2m}(a,b,q)}$ is closable and its closure in $L^2(\Omega)$, denoted by $Q_{A_{F,\Omega,2m}(a,b,q)}$, is the form uniquely associated to the Friedrichs extension $A_{F,\Omega,2m}(a,b,q)$ of $A_{\Omega,2m}(a,b,q)$, that is,*

$$Q_{A_{F,\Omega,2m}(a,b,q)}(f,g) = \left(A_{F,\Omega,2m}(a,b,q)^{1/2}f, A_{F,\Omega,2m}(a,b,q)^{1/2}g\right)_{L^2(\Omega)},$$
$$f,g \in \text{dom}\left(Q_{A_{F,\Omega,2m}(a,b,q)}\right) = \text{dom}\left(A_{F,\Omega,2m}(a,b,q)^{1/2}\right) = W_0^{m,2}(\Omega). \quad (3.47)$$

Proof. That $Q_{A_{\Omega,2m}(a,b,q)}$ is closable follows from abstract results relating sectorial (in particular, non-negative, symmetric) operators and their forms (cf., e.g., [28, Theorem IV.2.3], [54, Theorem VI.1.27], [74, Theorem X.23]). In order to prove (3.47), we fix $f \in W_0^{2m,2}(\Omega)$ and denote its extension by zero outside of Ω by $\tilde{f}$. Then $\tilde{f} \in W^{2m,2}(\mathbb{R}^n)$ and employing (3.28) with u replaced by $\tilde{f}$, and using the fact that $\text{supp}(\tilde{u}) \subseteq \Omega$, one obtains

$$c\|f\|^2_{W_0^{m,2}(\Omega)} \leqslant (f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \leqslant C\|f\|^2_{W_0^{m,2}(\Omega)}, \quad (3.48)$$

that is,

$$c\|f\|^2_{W_0^{m,2}(\Omega)} \leqslant Q_{A_{\Omega,2m}(a,b,q)}(f,f) + \|f\|^2_{L^2(\Omega)} \leqslant C\|f\|^2_{W_0^{m,2}(\Omega)}, \quad (3.49)$$

for some constants $0 < c < C$, proving that the domain of the closure of the form $Q_{A_{\Omega,2m}(a,b,q)}$ equals $W_0^{m,2}(\Omega)$. Together with [54, Sect. VI.2.3] or [74, Theorem X.23], and the second representation theorem for forms (see, e.g., [28, Theorem IV.2.6, Theorem IV.2.8], [54, Theorem VI.2.23]), this proves (3.47). $\square$

In Section 4 we will also use the following explicit representation of the form $Q_{A_{F,\Omega,2m}(a,b,q)}$,

$$Q_{A_{F,\Omega,2m}(a,b,q)}(f,g) = \begin{cases} (\tau_{2\ell}f, \tau_{2\ell}g)_{L^2(\Omega)}, & m = 2\ell, \ell \in \mathbb{N}, \\ \sum_{j,k=1}^n ((-i\partial_j - b_j)\tau_{2\ell}f, a_{j,k}(-i\partial_k - b_k)\tau_{2\ell}g)_{L^2(\Omega)} \\ +(\tau_{2\ell}f, q\,\tau_{2\ell}g)_{L^2(\Omega)}, & m = 2\ell + 1, \ k \in \mathbb{N} \cup \{0\}, \end{cases}$$
$$f, g \in W_0^{m,2}(\Omega). \quad (3.50)$$

(Again, we use the convention $\tau_0 = 1$.)

Finally, we introduce the following symmetric form in $L^2(\Omega)$,

$$\mathfrak{a}_{\Omega,4m,a,b,q}(f,g) := (A_{\Omega,2m}(a,b,q)f, A_{\Omega,2m}(a,b,q)g)_{L^2(\Omega)},$$
$$f, g \in \operatorname{dom}(\mathfrak{a}_{\Omega,4m,a,b,q}) := \operatorname{dom}(A_{\Omega,2m}(a,b,q)), \quad (3.51)$$

and the Hilbert space

$$\mathcal{H}_{A_{\Omega,2m}(a,b,q)} := (\operatorname{dom}(A_{\Omega,2m}(a,b,q)), \mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,))$$
$$= (W_0^{2m,2}(\Omega), \mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,)), \quad (3.52)$$

equipped with the scalar product $\mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\,\cdot\,)$.

Lemma 3.6. *Assume Hypothesis* 3.1 *(i). Then the Hilbert space* $\mathcal{H}_{A_{\Omega,2m}(a,b,q)}$ *embeds compactly into* $L^2(\Omega)$.

Proof. This is a consequence of the compact embedding of $\mathring{W}^{2m,2}$ into $L^2(\Omega)$ (see, e.g., [28, Theorem V.4.18]) and the inequalities (3.42). $\quad\square$

At this point we strengthen the lower bounds (3.39), (3.41):

Theorem 3.7. *Assume Hypothesis* 3.1 *(i) with* $m = 1$. *Then there exists* $\varepsilon > 0$, *only depending on* a *and* Ω, *such that* $A_{min,\Omega,2}(a,b,q)$ *defined as in* (3.38) *with* $m = 1$ *satisfies*

$$A_{min,\Omega,2}(a,b,q) \geqslant \varepsilon I_\Omega, \quad (3.53)$$

and hence,

$$A_{\Omega,2}(a,b,q) \geqslant \varepsilon I_\Omega \ \ and \ \ A_{F,\Omega,2}(a,b,q) \geqslant \varepsilon I_\Omega. \quad (3.54)$$

Proof. It suffices to derive $A_{F,\Omega,2}(a,b,q) \geqslant \varepsilon I_\Omega$. Since $\operatorname{dom}\big(A_{F,\Omega,2}(a,b,q)^{1/2}\big) = W_0^{1,2}(\Omega)$ according to (3.47), one recalls that

$$f \in W_0^{1,2}(\Omega) \text{ implies } |f| \in W_0^{1,2}(\Omega) \quad (3.55)$$

(cf., e.g., [28, Corollary VI.2.4]), and that by [72, Proposition 4.4],

$$\partial_j |f| = \operatorname{Re}\big(\operatorname{sgn}\big(\overline{f}\big)(\partial_j f)\big) \text{ a.e.}, \quad f \in W_0^{1,2}(\Omega), \ 1 \leqslant j \leqslant n, \quad (3.56)$$

with

$$\operatorname{sgn}(g(x)) = \begin{cases} g(x)/|g(x)|, & \text{if } g(x) \neq 0, \\ 0, & \text{if } g(x) = 0. \end{cases} \quad (3.57)$$

Thus, $\nabla|f| = \operatorname{Re}\big(\operatorname{sgn}\big(\overline{f}\big)(\nabla f)\big)$, $f \in W_0^{1,2}(\Omega)$, and hence one obtains the diamagnetic inequality on Ω,

$$|\nabla|f|| \leqslant \big|\operatorname{Re}\big(\operatorname{sgn}\big(\overline{f}\big)(\nabla f)\big)\big| = \big|\operatorname{Re}\big(\operatorname{sgn}\big(\overline{f}\big)((\nabla - ib)f)\big)\big| \leqslant |(-i\nabla - b)f| \text{ a.e.},$$
$$f \in W_0^{1,2}(\Omega), \quad (3.58)$$

since b_j, $1 \leqslant j \leqslant n$, are real-valued, according to a device of Kato [53] and Simon [88] (see also [8, Theorem 4.5.1], [65, Theorem 7.21]). Hence, employing the min-max principle for the infimum of the spectrum of self-adjoint operators bounded from below one estimates,

$$
\begin{aligned}
\inf(\sigma(A_{F,\Omega,2}(a,b,q))) &= \inf_{f \in W_0^{1,2}(\Omega), \, \|f\|_{L^2(\Omega)}=1} Q_{A_{F,\Omega,2}(a,b,q)}(f,f) \\
&= \inf_{f \in W_0^{1,2}(\Omega), \, \|f\|_{L^2(\Omega)}=1} \left(A_{F,\Omega,2}(a,b,q)^{1/2}f, A_{F,\Omega,2}(a,b,q)^{1/2}f\right)_{L^2(\Omega)} \\
&= \inf_{f \in W_0^{1,2}(\Omega), \, \|f\|_{L^2(\Omega)}=1} \left(\sum_{j,k=1}^n ((-i\partial_j - b_j)f, a_{j,k}(-i\partial_k - b_k)f)_{L^2(\Omega)} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + (f, q\,f)_{L^2(\Omega)} \right) \\
&\geqslant \varepsilon_a \inf_{f \in W_0^{1,2}(\Omega), \, \|f\|_{L^2(\Omega)}=1} ((-i\nabla - b)f, (-i\nabla - b)f)_{L^2(\Omega)^n} \\
&\geqslant \varepsilon_a \inf_{f \in W_0^{1,2}(\Omega), \, \|f\|_{L^2(\Omega)}=1} (|\nabla|f||, |\nabla|f||)_{L^2(\Omega)} \\
&= \varepsilon_a \inf_{f \in W_0^{1,2}(\Omega), \, \|f\|_{L^2(\Omega)}=1} (\nabla|f|, \nabla|f|)_{L^2(\Omega)^n} \\
&\geqslant \varepsilon_a \inf_{\varphi \in W_0^{1,2}(\Omega), \, \|\varphi\|_{L^2(\Omega)}=1} \left((\nabla\varphi, \nabla\varphi)_{L^2(\Omega)^n}\right) \\
&\geqslant \varepsilon_a \inf\left(\sigma\left(-\Delta_\Omega^D\right)\right) \\
&= \varepsilon_a \varepsilon_\Omega := \varepsilon,
\end{aligned}
\tag{3.59}
$$

using the fact that $-\Delta_\Omega^D \geqslant \varepsilon_\Omega I_\Omega$ for some $\varepsilon_\Omega > 0$, since Ω is bounded (see, for instance, [26, p. 31], or use domain monotonicity, [76, p. 270] together with the well-known strictly positive lower bounds for a ball or cube that encloses Ω). $\qquad\square$

The result (3.54) holds under more general assumptions on the coeffcients a, b, q and also for certain boundary conditions other than Dirichlet, but the current setup suffices for our discussion in the next Section 4 (we intend to revisit this issue elsewhere).

Next, we note that as a consequence of Hypothesis 3.1 (i) also all higher-order powers $A_{\Omega,2m}(a,b,q) = A_{\Omega,2}(a,b,q)^m$, $m \in \mathbb{N}$, $m \geqslant 2$, of $A_{\Omega,2}(a,b,q)$ are strictly positive.

Lemma 3.8. *Assume Hypothesis 3.1 (i). Then there exists $\varepsilon_m > 0$ such that*

$$
A_{\Omega,2m}(a,b,q) \geqslant \varepsilon_m I_\Omega, \quad m \in \mathbb{N}.
\tag{3.60}
$$

Proof. We employ induction with respect to $m \in \mathbb{N}$. The case $m = 1$ holds by Hypothesis 3.1 (i). Assume that the statement holds for all $k < m$ and fix any $0 \neq f \in \operatorname{dom}(A_{\Omega,2m}(a,b,q))$. We consider two cases:

(i) $m = 2\ell$, $\ell \in \mathbb{N}$. Then due to symmetry of $A_{\Omega,2}(a,b,q)^\ell$ one obtains

$$
(f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} = \left(f, A_{\Omega,2}(a,b,q)^{2\ell}f\right)_{L^2(\Omega)} = \left\|A_{\Omega,2}(a,b,q)^\ell f\right\|_{L^2(\Omega)}^2.
\tag{3.61}
$$

By the induction hypothesis, $A_{\Omega,2\ell}(a,b,q) \geqslant \varepsilon_\ell$, and hence

$$
\varepsilon_\ell \|f\|_{L^2(\Omega)}^2 \leqslant (f, A_{\Omega,2\ell}(a,b,q)f)_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega)} \|A_{\Omega,2\ell}(a,b,q)f\|_{L^2(\Omega)}, \tag{3.62}
$$

implying

$$(f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} = \left\|A_{\Omega,2}(a,b,q)^\ell f\right\|^2_{L^2(\Omega)} \geqslant \varepsilon_\ell^2 \|f\|^2_{L^2(\Omega)} = \varepsilon_m \|f\|^2_{L^2(\Omega)}, \tag{3.63}$$

with $\varepsilon_m = \varepsilon_\ell^2$.

(ii) $m = 2\ell + 1$, $\ell \in \mathbb{N}$. Then by (3.53)

$$\begin{aligned}
(f, A_{\Omega,2m}(a,b,q)f)_{L^2(\Omega)} &= \left(f, A_{\Omega,2}(a,b,q)^{2\ell+1}f\right)_{L^2(\Omega)} \\
&= \left(A_{\Omega,2}(a,b,q)^\ell f, A_{\Omega,2}(a,b,q)A_{\Omega,2}(a,b,q)^\ell f\right)_{L^2(\Omega)} \\
&\geqslant \varepsilon\left\|A_{\Omega,2}(a,b,q)^\ell f\right\|^2_{L^2(\Omega)} \geqslant \varepsilon\varepsilon_\ell^2 \|f\|^2_{L^2(\Omega)} = \varepsilon_m \|f\|^2_{L^2(\Omega)}, \tag{3.64}
\end{aligned}$$

with $\varepsilon_m = \varepsilon\varepsilon_\ell^2$. □

4. An Upper Bound for the Eigenvalue Counting Function for the Krein–von Neumann and Friedrichs Extensions of Higher-Order Operators

In this section we derive an upper bound for the eigenvalue counting function for Krein–von Neumann extensions of higher-order differential operators on open, bounded, nonempty domains $\Omega \subset \mathbb{R}^n$. In particular, no assumptions on the boundary of Ω will be made.

In the following we denote by $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$ the Krein–von Neumann and Friedrichs extensions of $A_{\Omega,2m}(a,b,q)$ in $L^2(\Omega)$. Since by Lemma 3.6, $\mathcal{H}_{A_{\Omega,2m}(a,b,q)}$ embeds compactly into $L^2(\Omega)$, $A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)$ has purely discrete spectrum by Lemma 2.10. Equivalently, $A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)$ has a compact resolvent, in particular,

$$[A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)]^{-1} \in \mathcal{B}_\infty\big(L^2(\Omega)\big). \tag{4.1}$$

Consequenty, also

$$|A_{\Omega,2m}(a,b,q)|^{-1} = [A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)]^{-1/2} \in \mathcal{B}_\infty\big(L^2(\Omega)\big), \tag{4.2}$$

implying

$$\big(\widehat{A}_{K,\Omega,2m}(a,b,q)\big)^{-1} \in \mathcal{B}_\infty\big(L^2(\Omega)\big) \tag{4.3}$$

by (2.43). Thus,

$$\sigma_{ess}(A_{K,\Omega,2m}(a,b,q)) \subseteq \{0\}. \tag{4.4}$$

We recall that the form $\mathfrak{a}_{\Omega,4m,a,b,q}(\,\cdot\,,\cdot\,)$ in $L^2(\Omega)$ associated with the operator $A_{\Omega,2m}(a,b,q)^* A_{\Omega,2m}(a,b,q)$ has been introduced in (3.51).

Let $\{\lambda_{K,\Omega,j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ be the strictly positive eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ enumerated in nondecreasing order, counting multiplicity, and let

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) := \#\{j \in \mathbb{N} \,|\, 0 < \lambda_{K,\Omega,j} < \lambda\}, \quad \lambda > 0, \tag{4.5}$$

be the eigenvalue distribution function for $A_{K,\Omega,2m}(a,b,q)$.

To derive an effective estimate for $N(\lambda; A_{K,\Omega,2m}(a,b,q))$ we need to introduce one more spectral hypothesis imposed on $\widetilde{A}_{2m}(a,b,q)$:

Hypothesis 4.1. *Assume Hypothesis* 3.1.
(i) *Suppose there exists* $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, *such that the operator*

$$(\mathbb{F}f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\overline{\phi(x,\xi)}\, d^n x, \quad \xi \in \mathbb{R}^n, \tag{4.6}$$

originally defined on functions $f \in L^2(\mathbb{R}^n)$ with compact support, can be extended to a unitary operator in $L^2(\mathbb{R}^n)$, such that

$$f \in W^{2,2}(\mathbb{R}^n; d^n x) \ \text{ if and only if } \ |\xi|^2 (\mathbb{F}f)(\xi) \in L^2(\mathbb{R}^n; d^n \xi), \qquad (4.7)$$

and

$$\widetilde{A}_2(a, b, q) = \mathbb{F}^{-1} M_{|\xi|^2} \mathbb{F}, \qquad (4.8)$$

where $M_{|\xi|^2}$ represents the maximally defined operator of multiplication by $|\xi|^2$ in $L^2(\mathbb{R}^n; d^n \xi)$.

(ii) In addition, suppose that

$$\sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|_{L^2(\Omega)} < \infty. \qquad (4.9)$$

Remark 4.2. *(i)* As becomes clear from Theorems 4.3 and 4.4 below, our primary concerns are the operators $A_{K,\Omega,2m}(a, b, q)$ and $A_{F,\Omega,2m}(a, b, q)$ in $L^2(\Omega)$, and hence we are primarily interested in the coefficients a, b, q on the open, bounded, but otherwise arbitrary, set Ω. However, since the existence of an eigenfunction expansion of a self-adjoint "continuation" of this pair of operators to all of $\mathbb{R}^n$, denoted by $\widetilde{A}_{2m}(a, b, q)$, is a crucial tool in our derivation of the bound on the corresponding eigenvalue counting functions of $A_{K,\Omega,2m}(a, b, q)$ and $A_{F,\Omega,2m}(a, b, q)$, the continuation of the coefficients a, b, q through a possibly highly nontrivial boundary $\partial\Omega$ of Ω becomes a nontrivial issue. To avoid intricate technicalities, we chose to simply assume a sufficiently smooth behavior of a, b, q throughout $\mathbb{R}^n$ in Hypothesis 3.1 *(i)*.

(ii) Hypothesis 4.1 *(i)* implies that $\widetilde{A}_2(a, b, q)$ (and hence any of its powers) is spectrally purely absolutely continuous (i.e., its point and singular continuous spectra are empty), while Hypothesis 4.1 *(ii)* requires a uniform $L^2(\Omega)$-bound on $\phi(\,\cdot\,, \xi)$, $\xi \in \mathbb{R}^n$. In particular, $\phi(\,\cdot\,, \,\cdot\,)$ represent the suitably normalized generalized eigenfunctions of $\widetilde{A}_2(a, b, q)$ satisfying

$$\widetilde{A}_2(a, b, q)\phi(\,\cdot\,, \xi) = |\xi|^2 \phi(\,\cdot\,, \xi), \quad \xi \in \mathbb{R}^n, \qquad (4.10)$$

in the distributional sense. In the special Laplacian case $a = I_n$, $b = q = 0$, one obtains

$$\phi(x, \xi) = e^{i\xi \cdot x}, \quad \|\phi(\,\cdot\,, \xi)\|^2_{L^2(\Omega)} = |\Omega|, \quad (x, \xi) \in \mathbb{R}^{2n}. \qquad (4.11)$$

(iii) In the case $a = I_n$, and with the exception of a possible zero-energy resonance and/or eigenvalue of $\widetilde{A}_2(I_n, b, q)$ in $L^2(\mathbb{R}^n)$, we expect Hypothesis 4.1 to hold for $\widetilde{A}_2(I_n, b, q)$ under the regularity assumptions made on b, q in Hypothesis 3.1 *(i)* assuming in addition that b and q have sufficiently fast decay as $|x| \to \infty$ (e.g., if b, q have compact support). While we have not found the corresponding statement in the literature, and an attempt to prove it in full generality would be an independent project, we will illustrate in our final Section 5 explicit situations in which Hypothesis 4.1 holds for $a = I_n$. The case $a \neq I_n$, on the other hand, is much more involved due trapping/non-trapping issues which affect the existence of bounds of the type (5.25); we refer, for instance, to [22], [23], [25], [93], [94], and the literature therein.

(iv) We note from the outset, that a zero-energy resonance and/or eigenvalue of $\widetilde{A}_{2m}$ cannot be excluded even in the special case $a = I_n$, $b = 0$, and $q \in C_0^\infty(\mathbb{R}^n)$. However, the existence of such zero-energy resonances or eigenvalues is highly unstable with respect to small variations of a, b, q and hence their absence holds generically. In particular, by slightly varying $R_0 > 0$ in Hypothesis 3.1 *(ii)*, or

the ε-neighborhood Ω_ε of Ω mentioned after (3.5), or by slightly perturbing the coefficients a, b, or q outside $B_n(0; R_0)$, or outside Ω_ε, one can guarantee the absence of such zero-energy resonances and/or eigenvalues. Since we are primarily interested in the operators $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$ in $L^2(\Omega)$, we can indeed freely choose the form of a, b, q in an ε-neighborhood outside of Ω, especially, in a neighborhood of infinity. $\diamond$

Recalling the standard notation

$$x_+ := \max(0, x), \quad x \in \mathbb{R}, \tag{4.12}$$

then $N(\,\cdot\,; A_{K,\Omega,2m}(a,b,q))$ permits the following estimate (extending the approach in [61] in the special case $a = I_n$, $b = q = 0$):

Theorem 4.3. *Assume Hypothesis* 4.1. *Then the following estimate holds:*

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant \frac{v_n}{(2\pi)^n}\left(1 + \frac{2m}{2m+n}\right)^{n/(2m)} \sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|^2_{L^2(\Omega)} \lambda^{n/(2m)},$$
$$\lambda > 0, \tag{4.13}$$

where $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ denotes the (Euclidean) volume of the unit ball in $\mathbb{R}^n$ ($\Gamma(\cdot)$ being the Gamma function), and $\phi(\,\cdot\,, \,\cdot\,)$ represents the suitably normalized generalized eigenfunctions of $\widetilde{A}_2(a,b,q)$ satisfying $\widetilde{A}_2(a,b,q)\phi(\,\cdot\,, \xi) = |\xi|^2\phi(\,\cdot\,, \xi)$ in the distributional sense (cf. Hypothesis 4.1).

Proof. Following our abstract Section 2, we introduce in addition to the symmetric form $\mathfrak{a}_{\Omega,4m,a,b,q}$ in $L^2(\Omega)$ (cf. (3.51)), the form

$$\mathfrak{b}_{\Omega,2m,a,b,q}(f,g) := (f, A_{\Omega,2m}(a,b,q)g)_{L^2(\Omega)},$$
$$f, g \in \mathrm{dom}(\mathfrak{b}_{\Omega,2m,a,b,q}) := \mathrm{dom}(A_{\Omega,2m}(a,b,q)). \tag{4.14}$$

By Lemma 2.7, particularly, by (2.49), one concludes that

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant \max\big(\dim\big\{f \in \mathrm{dom}(A_{\Omega,2m}(a,b,q))\,\big|$$
$$\mathfrak{a}_{\Omega,4m,a,b,q}(f,f) - \lambda\,\mathfrak{b}_{\Omega,2m,a,b,q}(f,f) < 0\big\}\big), \tag{4.15}$$

by also employing (2.51) and the fact that

$$\mathfrak{a}_{\Omega,4m,a,b,q}(f_{K,\Omega,j}, f_{K,\Omega,j})) - \lambda\,\mathfrak{b}_{\Omega,2m,a,b,q}(f_{K,\Omega,j}, f_{K,\Omega,j})$$
$$= (\lambda_{K,\Omega,j} - \lambda)\|f_{K,\Omega,j}\|^2_{L^2(\Omega)} < 0, \tag{4.16}$$

where $f_{K,\Omega,j} \in \mathrm{dom}(A_{\Omega,2m}(a,b,q))\backslash\{0\}$ additionally satisfies

$$f_{K,\Omega,j} \in \mathrm{dom}(A_{\Omega,2m}(a,b,q)^*A_{\Omega,2m}(a,b,q)) \text{ and}$$
$$A_{\Omega,2m}(a,b,q)^*A_{\Omega,2m}(a,b,q)f_{K,\Omega,j} = \lambda_{K,\Omega,j}\,A_{\Omega,2m}(a,b,q)f_{K,\Omega,j}. \tag{4.17}$$

To further analyze (4.15) we now fix $\lambda \in (0, \infty)$ and introduce the auxiliary operator

$$L_{\Omega,4m,\lambda}(a,b,q) := A_{\Omega,2m}(a,b,q)^*A_{\Omega,2m}(a,b,q) - \lambda\,A_{\Omega,2m}(a,b,q),$$
$$\mathrm{dom}(L_{\Omega,4m,\lambda}(a,b,q)) := \mathrm{dom}(A_{\Omega,2m}(a,b,q)^*A_{\Omega,2m}(a,b,q)). \tag{4.18}$$

By Lemma 2.8, $L_{\Omega,4m,\lambda}(a,b,q)$ is self-adjoint, bounded from below, with purely discrete spectrum as its form domain

$$\mathrm{dom}\big(|L_{\Omega,4m,\lambda}(a,b,q)|^{1/2}\big) = \mathcal{H}_{A_{\Omega,2m}(a,b,q)} \tag{4.19}$$

embeds compactly into $L^2(\Omega)$ by Lemma 3.6 (cf. Lemma 2.10). We will study the auxiliary eigenvalue problem,

$$L_{\Omega,4m,\lambda}(a,b,q)\varphi_j = \mu_j\varphi_j, \quad \varphi_j \in \operatorname{dom}(L_{\Omega,4m,\lambda}(a,b,q)), \tag{4.20}$$

where $\{\varphi_j\}_{j\in\mathbb{N}}$ represents an orthonormal basis of eigenfunctions in $L^2(\Omega)$ and for simplicity of notation we repeat the eigenvalues μ_j of $L_{\Omega,4m,\lambda}(a,b,q)$ according to their multiplicity. Since $\varphi_j \in W_0^{2m,2}(\Omega)$, we denote by

$$\widetilde{\varphi}_j(x) := \begin{cases} \varphi_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n\backslash\Omega, \end{cases} \tag{4.21}$$

the zero-extension of φ_j to all of $\mathbb{R}^n$ and observe that

$$\widetilde{\varphi}_j \in W^{2m,2}(\mathbb{R}^n), \quad \partial^\alpha\widetilde{\varphi}_j = \widetilde{\partial^\alpha\varphi_j}, \quad 0 \leqslant |\alpha| \leqslant 2m. \tag{4.22}$$

Next, given $\mu > 0$, one estimates

$$\mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}} (\mu-\mu_j) \geqslant \mu^{-1}\sum_{\substack{j\in\mathbb{N},\\ \mu_j<0,\,\mu_j<\mu}} (\mu-\mu_j) \geqslant \mu^{-1}\sum_{\substack{j\in\mathbb{N},\\ \mu_j<0,\,\mu_j<\mu}} \mu$$
$$= n_-(L_{\Omega,4m,\lambda}(a,b,q)), \tag{4.23}$$

where $n_-(L_{\Omega,4m,\lambda}(a,b,q))$ denotes the number of strictly negative eigenvalues of $L_{\Omega,4m,\lambda}(a,b,q)$. Combining, Lemma 2.9 and (4.15) one concludes that

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant \max\big(\dim\big\{f \in \operatorname{dom}(A_{min,\Omega,2m})\,\big|$$
$$\mathfrak{a}_{\Omega,4m,a,b,q}(f,f) - \lambda\,\mathfrak{b}_{\Omega,2m,a,b,q}(f,f) < 0\big\}\big) \tag{4.24}$$
$$= n_-(L_{\Omega,4m,\lambda}(a,b,q)) \leqslant \mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}} (\mu-\mu_j) = \mu^{-1}\sum_{j\in\mathbb{N}}[\mu-\mu_j]_+, \quad \mu > 0.$$

Next, we focus on estimating the right-hand side of (4.24).

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant \mu^{-1}\sum_{j\in\mathbb{N}}(\mu-\mu_j)_+ = \mu^{-1}\sum_{j\in\mathbb{N}}\big[(\varphi_j, (\mu-\mu_j)\varphi_j)_{L^2(\Omega)}\big]_+$$

$$= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\varphi_j\|_{L^2(\Omega)}^2 - \|A_{\Omega,2m}(a,b,q)\varphi_j\|_{L^2(\Omega)}^2$$
$$+ \lambda(\varphi_j, A_{\Omega,2m}(a,b,q)\varphi_j)_{L^2(\Omega)}\Big]_+$$

$$= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\widetilde{\varphi}_j\|_{L^2(\mathbb{R}^n)}^2 - \big\|\widetilde{A}_{2m}(a,b,q)\widetilde{\varphi}_j\big\|_{L^2(\mathbb{R}^n)}^2$$
$$+ \lambda\big(\widetilde{\varphi}_j, \widetilde{A}_{2m}(a,b,q)\widetilde{\varphi}_j\big)_{L^2(\mathbb{R}^n)}\Big]_+$$

$$= \mu^{-1}\sum_{j\in\mathbb{N}}\bigg[\int_{\mathbb{R}^n}\big[\mu-|\xi|^{4m}+\lambda|\xi|^{2m}\big]|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi\bigg]_+$$

$$\leqslant \mu^{-1}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^n}\big[\mu-|\xi|^{4m}+\lambda|\xi|^{2m}\big]_+|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi$$

$$\leqslant \mu^{-1}\int_{\mathbb{R}^n}\big[\mu-|\xi|^{4m}+\lambda|\xi|^{2m}\big]_+\sum_{j\in\mathbb{N}}|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi. \tag{4.25}$$

Since Ω is bounded, $\widetilde{\varphi}_j$ has compact support and hence

$$(\mathbb{F}\widetilde{\varphi}_j)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widetilde{\varphi}_j(x)\overline{\phi(x,\xi)}\, d^n x, \tag{4.26}$$

and

$$\sum_{j\in\mathbb{N}} |(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2 = (2\pi)^{-n} \sum_{j\in\mathbb{N}} \left| \int_{\mathbb{R}^n} \widetilde{\varphi}_j(x)\overline{\phi(x,\xi)}\, d^n x \right|^2$$

$$= (2\pi)^{-n} \sum_{j\in\mathbb{N}} \left| \int_{\Omega} \varphi_j(x)\overline{\phi(x,\xi)}\, d^n x \right|^2 = (2\pi)^{-n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega,d^n x)}, \tag{4.27}$$

are well-defined. Combining (4.25) and (4.27) one arrives at

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant \mu^{-1} \int_{\mathbb{R}^n} \left[\mu - |\xi|^{4m} + \lambda|\xi|^{2m} \right]_+ \sum_{j\in\mathbb{N}} |(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\, d^n\xi$$

$$= (2\pi)^{-n}\mu^{-1} \int_{\mathbb{R}^n} \left[\mu - |\xi|^{4m} + \lambda|\xi|^{2m} \right]_+ \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\, d^n\xi$$

$$\leqslant (2\pi)^{-n} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\mu^{-1} \int_{\mathbb{R}^n} \left[\mu - |\xi|^{4m} + \lambda|\xi|^{2m} \right]_+ d^n\xi. \tag{4.28}$$

Introducing $\alpha = \lambda^{-2}\mu$, changing variables, $\xi = \lambda^{1/(2m)}\eta$, and taking the minimum with respect to $\alpha > 0$, proves the bound,

$$N(\lambda; A_{K,\Omega,2m}(a,b,q)) \leqslant (2\pi)^{-n} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}$$

$$\times \min_{\alpha>0} \left(\alpha^{-1} \int_{\mathbb{R}^n} \left[\alpha - |\eta|^{4m} + |\eta|^{2m} \right]_+ d^n\eta \right) \lambda^{n/(2m)}, \quad \lambda > 0. \tag{4.29}$$

Explicitly computing the minimum over $\alpha > 0$ in (4.29) yields the result (4.13). This minimization step is carried out in detail in Appendix A. $\qquad\square$

Next, we also derive an upper bound for the eigenvalue counting function of the Friedrichs extension $A_{F,\Omega,2m}(a,b,q)$ of $A_{\Omega,2m}(a,b,q)$.

Theorem 4.4. *Assume Hypothesis* 4.1. *Then the following estimate holds:*

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leqslant \frac{v_n}{(2\pi)^n} \left(1 + \frac{2m}{n} \right)^{n/(2m)} \sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)}\, \lambda^{n/(2m)},$$

$$\lambda > 0, \quad (4.30)$$

with $v_n := \pi^{n/2}/\Gamma((n+2)/2)$ *and* $\phi(\,\cdot\,,\,\cdot\,)$ *given as in Theorem* 4.3.

Proof. First, one notices that

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leqslant \max\big(\dim \big\{ f \in \mathrm{dom}(A_{F,\Omega,2m}(a,b,q)) \,\big|$$

$$(f, A_{F,\Omega,2m}(a,b,q)f)_{L^2(\Omega)} - \lambda\|f\|^2_{L^2(\Omega)} < 0 \big\}\big), \tag{4.31}$$

To further analyze the right–hand side of (4.31) we now fix $\lambda \in (0,\infty)$ and introduce the auxiliary operator

$$K_{\Omega,2m,\lambda}(a,b,q) := A_{F,\Omega,2m}(a,b,q) - \lambda I_\Omega,$$

$$\mathrm{dom}(K_{\Omega,2m,\lambda}(a,b,q)) := \mathrm{dom}(A_{F,\Omega,2m}(a,b,q)). \tag{4.32}$$

We will study the eigenvalue problem,

$$K_{\Omega,2m,\lambda}(a,b,q)\varphi_j = \mu_j\varphi_j, \quad \varphi_j \in \operatorname{dom}(K_{\Omega,m,\lambda}(a,b,q)), \tag{4.33}$$

where $\{\varphi_j\}_{j\in\mathbb{N}}$ represents an orthonormal basis of eigenfunctions in $L^2(\Omega)$ and for simplicity of notation we repeat the eigenvalues μ_j of $K_{\Omega,2m,\lambda}(a,b,q)$ according to their multiplicity. Since $\varphi_j \in W_0^m(\Omega)$, we denote by

$$\widetilde{\varphi}_j(x) := \begin{cases} \varphi_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n\backslash\Omega, \end{cases} \tag{4.34}$$

their zero-extension to all of $\mathbb{R}^n$ and note that

$$\widetilde{\varphi}_j \in W^m(\mathbb{R}^n), \quad \partial^\alpha\widetilde{\varphi}_j = \widetilde{\partial^\alpha\varphi_j}, \quad 0 \leqslant |\alpha| \leqslant m. \tag{4.35}$$

Next, given $\mu > 0$, one estimates

$$\mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}}(\mu-\mu_j) \geqslant \mu^{-1}\sum_{\substack{j\in\mathbb{N},\\ \mu_j<0,\,\mu_j<\mu}}(\mu-\mu_j) \geqslant \mu^{-1}\sum_{\substack{j\in\mathbb{N},\\ \mu_j<0,\,\mu_j<\mu}}\mu$$

$$= n_-(K_{\Omega,2m,\lambda}(a,b,q)), \tag{4.36}$$

where $n_-(K_{\Omega,2m,\lambda}(a,b,q))$ denotes the number of strictly negative eigenvalues of $K_{\Omega,2m,\lambda}(a,b,q)$. Then one has

$$N(\lambda; A_{F,\Omega,2m}(a,b,q))$$
$$\leqslant \max\big(\dim\big\{f \in \operatorname{dom}(A_{F,\Omega,2m}(a,b,q))\,\big|$$
$$(f, A_{F,\Omega,2m}(a,b,q)f)_{L^2(\Omega)} - \lambda\|f\|_{L^2(\Omega)}^2 < 0\big\}\big)$$
$$= n_-(K_{\Omega,2m,\lambda}(a,b,q)) \leqslant \mu^{-1}\sum_{\substack{j\in\mathbb{N}\\ \mu_j<\mu}}(\mu-\mu_j) = \mu^{-1}\sum_{j\in\mathbb{N}}[\mu-\mu_j]_+, \quad \mu > 0. \tag{4.37}$$

To estimate the right-hand side of (4.37) we rewrite $(\psi_1, A_{F,\Omega,2m}(a,b,q)\psi_2)_{L^2(\Omega)}$ for $\psi_1, \psi_2 \in \operatorname{dom}(A_{F,\Omega,2m}(a,b,q))$, as follows

$$(\psi_1, A_{F,\Omega,2m}(a,b,q)\psi_2)_{L^2(\Omega)} = Q_{A_{F,\Omega,2m}(a,b,q)}(\psi_1,\psi_2) = Q_{\widetilde{A}_{2m}(a,b,q)}(\widetilde{\psi}_1, \widetilde{\psi}_2)$$

$$= \Big(\big(\widetilde{A}_{2m}(a,b,q)\big)^{1/2}\widetilde{\psi}_1, \big(\widetilde{A}_{2m}(a,b,q)\big)^{1/2}\widetilde{\psi}_2\Big)_{L^2(\mathbb{R}^n)}, \tag{4.38}$$

the second equality in (4.38) following from representations (3.36), (3.50). Next, we focus on estimating the right-hand side of (4.37).

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leqslant \mu^{-1}\sum_{j\in\mathbb{N}}(\mu-\mu_j)_+ = \mu^{-1}\sum_{j\in\mathbb{N}}\big[(\varphi_j, (\mu-\mu_j)\varphi_j)_{L^2(\Omega)}\big]_+$$

$$= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\varphi_j\|_{L^2(\Omega)}^2 + \lambda\|\varphi_j\|_{L^2(\Omega)}^2 - (\varphi_j, A_{F,\Omega,2m}(a,b,q)\varphi_j)_{L^2(\Omega)}\Big]_+$$

$$= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\mu\|\mathbb{F}\widetilde{\varphi}_j\|_{L^2(\mathbb{R}^n)}^2 + \lambda\|\mathbb{F}\widetilde{\varphi}_j\|_{L^2(\mathbb{R}^n)}^2 - \||\xi|^m\mathbb{F}\widetilde{\varphi}_j\|_{L^2(\mathbb{R}^n)}^2\Big]_+$$

$$= \mu^{-1}\sum_{j\in\mathbb{N}}\Big[\int_{\mathbb{R}^n}\big[\mu + \lambda - |\xi|^{2m}\big]|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi\Big]_+$$

$$\leqslant \mu^{-1}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^n}\big[\mu + \lambda - |\xi|^{2m}\big]_+|(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2\,d^n\xi$$

$$\leqslant \mu^{-1} \int_{\mathbb{R}^n} \left[\mu + \lambda - |\xi|^{2m}\right]_+ \sum_{j \in \mathbb{N}} |(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2 \, d^n\xi. \tag{4.39}$$

Combining (4.27) and (4.39) one arrives at

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leqslant \mu^{-1} \int_{\mathbb{R}^n} \left[\mu + \lambda - |\xi|^{2m}\right]_+ \sum_{j \in \mathbb{N}} |(\mathbb{F}\widetilde{\varphi}_j)(\xi)|^2 \, d^n\xi$$

$$= (2\pi)^{-n} \mu^{-1} \int_{\mathbb{R}^n} \left[\mu + \lambda - |\xi|^{2m}\right]_+ \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)}^2 \, d^n\xi$$

$$\leqslant (2\pi)^{-n} \sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)}^2 \mu^{-1} \int_{\mathbb{R}^n} \left[\mu + \lambda - |\xi|^{2m}\right]_+ \, d^n\xi. \tag{4.40}$$

Introducing $\alpha = \lambda^{-1}\mu$, changing variables, $\xi = \lambda^{1/(2m)}\eta$, and taking the minimum with respect to $\alpha > 0$, proves the bound,

$$N(\lambda; A_{F,\Omega,2m}(a,b,q)) \leqslant (2\pi)^{-n} \sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)}^2$$
$$\times \min_{\alpha > 0} \left(\alpha^{-1} \int_{\mathbb{R}^n} \left[\alpha + 1 - |\eta|^{2m}\right]_+ d^n\eta\right) \lambda^{n/(2m)}, \quad \lambda > 0. \tag{4.41}$$

Denoting

$$\mathcal{I}_F(\alpha) := \alpha^{-1} \int_{\mathbb{R}^n} \left[\alpha + 1 - |\eta|^{2m}\right]_+ d^n\eta, \tag{4.42}$$

one explicitly computes $\mathcal{I}_F(\alpha)$ and obtains

$$\mathcal{I}_F(\alpha) = \frac{2m v_n}{2m + n} \alpha^{-1} (\alpha + 1)^{(2m+n)/(2m)}, \tag{4.43}$$

$$\mathcal{I}_F'(\alpha) = \frac{n v_n}{2m + n} (\alpha + 1)^{n/(2m)} \alpha^{-2} \left(\alpha - \frac{2m}{n}\right), \tag{4.44}$$

$$\min_{\alpha > 0} \left(\mathcal{I}_F(\alpha)\right) = \mathcal{I}_F(2m/n) = v_n \left(1 + \frac{2m}{n}\right)^{n/(2m)}. \tag{4.45}$$

Equation (4.45) together with (4.41) yields (4.30). $\qquad\square$

Remark 4.5. (*i*) One notes that whenever the property

$$\sup_{(x,\xi) \in \Omega \times \mathbb{R}^n} \left(|\phi(x,\xi)|\right) < \infty \tag{4.46}$$

has been established, then

$$\sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)}^2 \leqslant |\Omega| \sup_{(x,\xi) \in \Omega \times \mathbb{R}^n} \left(|\phi(x,\xi)|^2\right), \tag{4.47}$$

explicitly exhibits the volume dependence on Ω of the right-hand sides of (4.13) and (4.30), respectively. We will briefly revisit this in Section 5.

(*ii*) Given two self-adjoint operators A, B in $\mathcal{H}$ bounded from below with purely discrete spectra such that $A \leqslant B$ in the sense of quadratic forms, then clearly $N(\lambda; B) \leqslant N(\lambda; A)$, $\lambda \in \mathbb{R}$; in addition, $N(\lambda; \alpha A) = N(\lambda/\alpha; A)$, $\alpha > 0$, $\lambda \in \mathbb{R}$. Thus, since a is real symmetric, the uniform ellipticity condition (3.4) implies $a \geqslant \varepsilon_a I_n$, and hence $A_{F,\Omega,2}(a,b,q) \geqslant \varepsilon_a A_{F,\Omega,2}(I_n,b,q)$ assuming $\varepsilon_a \in (0,1]$ without loss of generality. Combining this with (2.24) then yields

$$N(\lambda; A_{K,\Omega,2}(a,b,q)) \leqslant N(\lambda; A_{F,\Omega,2}(a,b,q)) \leqslant N(\lambda; \varepsilon_a A_{F,\Omega,2}(I_n,b,q))$$
$$= N(\lambda/\varepsilon_a; A_{F,\Omega,2}(I_n,b,q)), \quad \lambda \in \mathbb{R}. \tag{4.48}$$

Finally, we note that estimates of the type $N(\lambda; A) \leqslant c_A \lambda^\gamma$ for $A \geqslant 0$ yield lower bounds for the jth eigenvalue $\lambda_j(A)$ of the form $\lambda_j(A) \geqslant d_A j^{1/\gamma}$, clearly applicable in the context of (4.13) and (4.30). ◇

Remark 4.6. As far as we know, employing the technique of the eigenfunction transform (i.e., the distorted Fourier transform) associated with the variable coefficient operator $\widetilde{A}_{2m}(a, b, q)$ (replacing the standard Fourier transform in connection with the constant coefficient case in [38]) to derive the results (1.12) and (1.13) is new.

On the other hand, the literature on eigenvalue counting function bounds in connection with arbitrary bounded open sets $\Omega \subset \mathbb{R}^n$ (or even open sets $\Omega \subset \mathbb{R}^n$ of finite Euclidean volume) is fairly extensive, originating with the seminal work by Birman–Solomyak, Rozenblum, and others. More specifically, starting around 1970, in this context of rough sets Ω, Birman and Solomyak pioneered the leading-order Weyl asymptotics and eigenvalue counting function estimates for generalized (linear pencil) eigenvalue problems of the form $Af = \lambda Bf$ for elliptic partial differential operators A of order n_A and lower-order differential operators B of order $n_B < n_A$ and obtained great generality of the coeffcients in A and B by systematically employing a variational formulation of this generalized eigenvalue problem. The boundary conditions employed are frequently of Dirichlet type, but Neumann and Robin boundary conditions are studied as well. In particular (focusing on the Dirichlet case only), the variational form of the problem associated with

$$\sum_{|\alpha|=|\beta|=m} D^\alpha \big(a_{\alpha,\beta}(x) D^\beta u\big)(x) = \lambda \, p(x) \, u(x), \quad u \in W_0^{m,2}(\Omega), \qquad (4.49)$$

with special emphasis on the polyharmonic case, $(-\Delta)^m u = \lambda \, p \, u$, and extensions to the situation

$$\sum_{|\alpha|=|\beta|=m} D^\alpha \big(a_{\alpha,\beta}(x) D^\beta u\big)(x) = \lambda \sum_{0 \leqslant |\gamma|, |\delta| \leqslant m} D^\gamma \big(b_{\gamma,\delta}(x) D^\delta u\big)(x),$$
$$|\gamma| + |\delta| = 2\ell, \, 0 \leqslant \ell < m, \, u \in W_0^{m,2}(\Omega), \qquad (4.50)$$

including the scenario where a, b are block matrices, or b is an appropriate (matrix-valued) measure, were studied in [12]–[18], [78]–[81], [82, Ch. 5]. In particular, the hypotheses on $a_{\alpha,\beta}$ are very general ($a \in L^1_{loc}(\Omega)^{m \times m}$, a positive definite a.e., $a^{-1} \in L^\alpha(\Omega)^{m \times m}$ for appropriate $\alpha \geqslant 1$) permitting a certain weak degeneracy of the ellipticity of the left-hand side in (4.49), (4.50). The case of the Friedrichs extension for $m = 1$ corresponding to $\tau_2(a, b, q)$ was treated in [67].

Thus, in the case $m = 1$, $p(\cdot) = 1$, and in some particular higher-order cases, where $m > 1$, in the context of $A_{F,\Omega,2m}(a, 0, 0)$ (i.e., $b = q = 0$), there is clearly some overlap of our result (4.30) with the above results concerning (4.49). The same applies to the magnetic field results in [67] in connection with $\tau_2(a, b, q)$. Similarly, considering the perturbed buckling problem in the form

$$(-\Delta)^{2m} u = \lambda \, (-\Delta)^m u, \quad u \in W_0^{2m,2}(\Omega), \qquad (4.51)$$

there is of course some overlap between our result (4.13) (actually, the result in [38]) and the results concerning (4.50) with $m \in \mathbb{N}$, $a = I_n$, $b = q = 0$, but since lower-order terms are not explicitly included on the left-hand side of (4.50), a direct comparison is difficult. According to G. Rozenblum (private communication), the left-hand sides in (4.49), (4.50) can be extended to include also lower-order

terms under appropriate hypotheses on the coefficients, but this seems not to have appeared explicitly in print.

Since we focused on the case of nonconstant coefficients throughout, we did not enter the vast literature on eigenvalue counting function estimates in connection with the Laplacian and its (fractional) powers. In this context we refer, for instance, to [34], [48], [97], [98], and the extensive literature cited therein.   ◇

Although Weyl asymptotics itself is not the objective of this paper, we conclude this section with the following observation.

Remark 4.7. The Weyl asymptotics of $N(\,\cdot\,;A_{K,\Omega,2}(a,b,q))$ in [5, Sect. 8] in the case of quasi-convex domains and in [9] in the case of bounded Lipschitz domains derived an error bound of the form $O\big(\lambda^{(n-(1/2))/2}\big)$ as $\lambda \to \infty$. If one is only interested in the leading-order asymptotics results, combining the spectral equivalence of nonzero eigenvalues of $A_{K,\Omega,2m}(a,b,q))$ to the (generalized) buckling problem (cf. Lemma 2.5), with results by Kozlov [56]–[58], and taking into account that lower-order differential operator perturbations do not influence the leading-order asymptotics of $N(\,\cdot\,;A_{K,\Omega,2m}(a,b,q))$ (cf. [15, Lemmas 1.3, 1.4]) imply

$$N(\lambda; A_{K,\Omega,2m}(a,b,q))$$

$$\underset{\lambda\to\infty}{=} \frac{1}{n(2\pi)^n}\left(\int_\Omega d^n x \int_{|\xi|=1} d\omega_{n-1}(\xi)(\xi,a(x)\,\xi)_{\mathbb{R}^n}^{-\frac{n}{2}}\right)\lambda^{n/(2m)} + o\big(\lambda^{n/(2m)}\big)$$

$$\underset{\lambda\to\infty}{=} \frac{v_n}{(2\pi)^n}\left(\int_\Omega d^n x\,(\det a(x))^{-1/2}\right)\lambda^{n/(2m)} + o\big(\lambda^{n/(2m)}\big), \qquad (4.52)$$

for any bounded open set $\Omega \subset \mathbb{R}^n$. Here $d\omega_{n-1}$ denotes the surface measure on the unit sphere $S^{n-1} = \{\xi \in \mathbb{R}^n \,|\, |\xi| = 1\}$ in $\mathbb{R}^n$. Of course, the same leading-order asymptotics applies to $N(\,\cdot\,;A_{F,\Omega,2m}(a,b,q))$.

Since $N(\lambda; A) \underset{\lambda\to\infty}{=} c(A)\lambda^\alpha$ is equivalent to $\lambda_j(A) \underset{j\to\infty}{=} (j/c(A))^{1/\alpha}$, relation (4.52) yields the corresponding result for the eigenvalues of $A_{K,\Omega,2m}(a,b,q)$ and $A_{F,\Omega,2m}(a,b,q)$.   ◇

5. Illustrations

To demonstrate why we expect Hypothesis 4.1 to hold under Hypothesis 3.1 alone in the case $a = I_n$ (with the obvious exception of zero-energy resonances and eigenvalues, which generically will be absent), we discuss three exceedingly complex scenarios in this section.

We start with the most elementary case which nevertheless served as the guiding motivation for this paper:

Example 5.1. *Let* $a := I_n$, $n \in \mathbb{N}$, $b = q = 0$, *then the operator* $\mathbb{F}$ *from Theorem 4.1 is the standard Fourier transform in* $L^2(\mathbb{R}^n)$, *and* $\phi(\xi,x) = e^{i\xi\cdot x}$, $(\xi,x) \in \mathbb{R}^{2n}$. *Thus, Hypothesis 4.1 obviously holds for* $\widetilde{A}_2(I_n,0,0) = H_0$, *and*

$$\sup_{\xi\in\mathbb{R}^n}\|\phi(\,\cdot\,,\xi)\|^2_{L^2(\Omega)} = |\Omega|. \qquad (5.1)$$

In this rather special case the estimate for the eigenvalue counting function $N(\lambda; -\Delta_{K,\Omega})$ was previously obtained in [38], while that of $N(\lambda; -\Delta_{D,\Omega})$ was derived in [61].

Next, we turn to Schrödinger operators in $L^2(\mathbb{R}^n)$.

Example 5.2. *Assume that $a = I_n$, $b = 0$, and $0 \leqslant q \in L^\infty(\mathbb{R}^n)$, $\mathrm{supp}(q)$ compact. In addition, suppose that zero is neither an eigenvalue nor a resonance of $\widetilde{A}_2(I_n, b, q)$ (cf. [30]). Then Hypothesis 4.1 holds.*

In addition, in the special case $n = 3$, there exists $C(q) \in (0, \infty)$ such that

$$\sup_{(x,\xi) \in \mathbb{R}^6} |\phi(x, \xi)| \leqslant C(q). \tag{5.2}$$

Indeed, the absence of strictly positive eigenvalues of $\widetilde{A}_2(I_n, 0, q)$ was established by Kato [52] (see also [86]), and the existence of the distorted Fourier transform $\mathbb{F}$ and hence an eigenfunction transform was established by Ikebe [49, Theorem 5] for $n = 3$ and Thoe [91, Sect. 4] for $n \geqslant 4$, and Alsholm and Schmidt [3] for $n \geqslant 3$ (see also [75, Theorem XI.41], [76, Theorems XIII.33 and XIII.58], [77], [87, Sect. V.4]), implying, in particular, that

$$\sigma\big(\widetilde{A}_2(I_n, 0, q)\big) = \sigma_{ac}\big(\widetilde{A}_2(I_n, 0, q)\big) = [0, \infty),$$
$$\sigma_{sc}\big(\widetilde{A}_2(I_n, 0, q)\big) = \sigma_p\big(\widetilde{A}_2(I_n, 0, q)\big) \cap (0, \infty) = \emptyset. \tag{5.3}$$

Moreover, it is shown in [49] and [91] that for all $R > 0$,

$$\sup_{\xi \in B_n(0;R),\, x \in \mathbb{R}^n} |\phi(x, \xi)| =: c(q, R) < \infty. \tag{5.4}$$

Thus we will focus on proving that

$$\sup_{\xi \in \mathbb{R}^n} \|\phi(\,\cdot\,, \xi)\|_{L^2(\Omega)} < \infty, \tag{5.5}$$

and in the special case $n = 3$ that for sufficiently large $R > 0$,

$$\sup_{\xi \in \mathbb{R}^3 \backslash B_3(0;R),\, x \in \mathbb{R}^3} |\phi(x, \xi)| =: C(q, R) < \infty. \tag{5.6}$$

Clearly, estimates (5.4) and (5.6) imply (5.2).

The distorted plane waves $\phi(\,\cdot\,, \,\cdot\,)$ can be chosen as one of $\phi_+(\,\cdot\,, \,\cdot\,)$ or $\phi_-(\,\cdot\,, \,\cdot\,)$, which are defined as solutions of the following Lippmann–Schwinger integral equation,

$$\phi_\pm(x, \xi) = e^{i\xi \cdot x} - \int_{\mathbb{R}^n} G_n\big(|\xi|^2 \pm i0; x, y\big) q(y) \phi_\pm(y, \xi)\, d^n y, \quad (x, \xi) \in \mathbb{R}^{2n}, \tag{5.7}$$

where

$$G_n(z; x, y) = \begin{cases} \frac{i}{4}\left(\frac{2\pi|x-y|}{z^{1/2}}\right)^{(2-n)/2} H^{(1)}_{(n-2)/2}\big(z^{1/2}|x - y|\big), & n \geqslant 2,\ z \in \mathbb{C} \backslash \{0\}, \\ \frac{-1}{2\pi} \ln(|x - y|), & n = 2,\ z = 0, \\ \frac{1}{(n-2)\omega_{n-1}} |x - y|^{2-n}, & n \geqslant 3,\ z = 0, \end{cases}$$
$$\mathrm{Im}\big(z^{1/2}\big) \geqslant 0,\ x, y \in \mathbb{R}^n,\ x \neq y, \tag{5.8}$$

represents the fundamental solution of the Helmholtz equation $(-\Delta - z)\psi(z; \,\cdot\,) = 0$ in $\mathbb{R}^n$, that is, the Green's function of the n-dimensional Laplacian, $n \in \mathbb{N}$, $n \geqslant 2$. Here $H^{(1)}_\nu(\,\cdot\,)$ denotes the Hankel function of the first kind with index $\nu \geqslant 0$ (cf. [1, Sect. 9.1]) and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ ($\Gamma(\,\cdot\,)$ the Gamma function, cf. [1, Sect. 6.1]) represents the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$. For simplicity we focus on $n \geqslant 3$ for the rest of this example, but note that the cases $n = 1, 2$ can be treated exactly along the same lines (see, e.g., the results in [19]–[24]).

Multiplying both sides of this equation by the weight $w > 0$ satisfying

$$w \in C^\infty(\mathbb{R}^n), \quad 0 < w \leqslant 1, \quad w(x) := \begin{cases} 1, & 0 \leqslant |x| \leqslant R, \\ \exp(-|x|^2), & |x| \geqslant 2R, \end{cases} \tag{5.9}$$

$$\Omega \subset B_n(0; R),$$

for some $R > 0$, (5.7) can be written as follows

$$\Phi_\pm(x, \xi) = \Phi_0(x, \xi) - \int_{\mathbb{R}^n} w(x) G_n(|\xi|^2 \pm i0; x, y) w(y) \frac{q(y)}{w^2(y)} \Phi_\pm(y, \xi) \, d^3 y, \tag{5.10}$$
$$(x, \xi) \in \mathbb{R}^{2n},$$

where

$$\Phi_\pm(x, \xi) := w(x) \phi_\pm(x, \xi), \quad \Phi_0(x, \xi) := w(x) e^{i\xi \cdot x}, \quad (x, \xi) \in \mathbb{R}^{2n}. \tag{5.11}$$

In this form (5.10) becomes an integral equation in $L^2(\mathbb{R}^n)$ since $\Phi_0(\,\cdot\,, \xi) \in L^2(\mathbb{R}^n)$. In fact, (5.10) will be viewed in $L^2(\mathbb{R}^n)$ as

$$\Phi_\pm(\,\cdot\,, \xi) = \Phi_0(\,\cdot\,, \xi) + K_\pm(\xi) M_{q/w^2} \Phi_\pm(\,\cdot\,, \xi), \quad \xi \in \mathbb{R}^n, \tag{5.12}$$

or equivalently, as

$$[I_{L^2(\mathbb{R}^n, d^n x)} - K_\pm(\xi) M_{q/w^2}] \Phi_\pm(\,\cdot\,, \xi) = \Phi_0(\,\cdot\,, \xi), \quad \xi \in \mathbb{R}^n, \tag{5.13}$$

where we introduced the Birman–Schwinger-type operator $K_\pm(\xi)$, $\xi \in \mathbb{R}^n$, in $L^2(\mathbb{R}^n)$,

$$K_\pm(\xi) \in \mathcal{B}(L^2(\mathbb{R}^n)),$$

$$(K_\pm(\xi) f)(x) := -\int_{\mathbb{R}^n} w(x) G_n(|\xi|^2 \pm i0; x, y) w(y) f(y, \xi) \, d^n y, \tag{5.14}$$
$$f \in L^2(\mathbb{R}^n), \ (x, \xi) \in \mathbb{R}^{2n},$$

and the operator of multiplication by the function q/w^2, M_{q/w^2} in $L^2(\mathbb{R}^n)$,

$$M_{q/w^2} \in \mathcal{B}(L^2(\mathbb{R}^n)), \quad (M_{q/w^2} f)(x) := q(x) w(x)^{-2} f(x), \quad f \in L^2(\mathbb{R}^n), \ x \in \mathbb{R}^n. \tag{5.15}$$

One recalls from [87, Sect. V.4] for $n = 3$ and [32] for $n \geqslant 3$ (the case $n = 2$ being analogous) that

$$\|K_\pm(\xi)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \underset{|\xi| \to \infty}{\longrightarrow} 0, \tag{5.16}$$

and hence,

$$\|\Phi_\pm(\,\cdot\,, \xi) - \Phi_0(\,\cdot\,, \xi)\|_{L^2(\mathbb{R}^n)}$$
$$= \left\| \left(I_{L^2(\mathbb{R}^n)} - (I_{L^2(\mathbb{R}^n)} - K_\pm(\xi) M_{q/w^2}))^{-1} \right) \Phi_0(\,\cdot\,, \xi) \right\|_{L^2(\mathbb{R}^n)}$$
$$\leqslant \|w(\cdot)\|_{L^2(\mathbb{R}^n)} \left\| I_{L^2(\mathbb{R}^n)} - (I_{L^2(\mathbb{R}^n)} - K_\pm(\xi) M_{q/w^2}))^{-1} \right\|_{\mathcal{B}(L^2(\mathbb{R}^n))}$$
$$\underset{|\xi| \to \infty}{=} o(1), \tag{5.17}$$

implying,

$$\|\Phi_\pm(\,\cdot\,, \xi)\|_{L^2(\mathbb{R}^n)} \underset{|\xi| \to \infty}{=} O(1), \tag{5.18}$$

and hence (5.5).

In the special case $n = 3$, where

$$G_3(z; x, y) = (4\pi |x - y|)^{-1} e^{iz^{1/2}|x-y|}, \quad \text{Im}(z^{1/2}) \geqslant 0, \ x, y \in \mathbb{R}^3, \ x \neq y, \tag{5.19}$$

one can easily go one step further: Using the Cauchy–Schwarz inequality, (5.18), and the fact that q has compact support, one estimates the second term in (5.10) as follows,

$$
\left| \int_{\mathbb{R}^3} \frac{w(x)e^{\pm i|\xi||x-y|}w(y)}{4\pi|x-y|} \frac{q(y)}{w^2(y)} \Phi_\pm(y,\xi)\, d^3y \right|
$$
$$
\leqslant (4\pi)^{-1} w(x) \int_{\mathrm{supp}(q)} \frac{w(y)}{|x-y|} \frac{q(y)}{w^2(y)} |\Phi_\pm(y,\xi)|\, d^3y
$$
$$
\leqslant (4\pi)^{-1} w(x) \|qw^{-2}\|_{L^\infty(\mathbb{R}^3)} \left(\int_{\mathrm{supp}(q)} \frac{w^2(y)}{|x-y|^2}\, d^n y \right)^{1/2} \|\Phi_\pm(\,\cdot\,,\xi)\|_{L^2(\mathbb{R}^3)}
$$
$$
\underset{|\xi|\to\infty}{=} w(x)O(1), \quad x \in \mathbb{R}^3, \tag{5.20}
$$

with the $O(1)$-term bounded uniformly in $(x,\xi) \in \mathbb{R}^6$. Combining (5.11), (5.12), and (5.20) one obtains

$$
\sup_{x\in\mathbb{R}^3} |\phi_\pm(x,\xi)| \underset{|\xi|\to\infty}{=} O(1), \tag{5.21}
$$

proving (5.4) since $\phi_\pm$ is continuous on $\mathbb{R}^6$ (see, e.g., [49, Sect. 4], [91, Sect. 3]). $\square$

Example 5.3. *Assume that $n \in \mathbb{N}$, $a = I_n$, $b \in \left[W^{1,\infty}(\mathbb{R}^n)\right]^n$, $\mathrm{supp}(b)$ compact, $0 \leqslant q \in L^\infty(\mathbb{R}^n)$, $\mathrm{supp}(q)$ compact. In addition, suppose that zero is neither an eigenvalue nor a resonance of $\widetilde{A}_2(I_n,b,q)$ (cf. [30]). Then Hypothesis 4.1 holds.*

We start verifying this claim by noting that under these assumptions on a, b, q, $\widetilde{A}_2(I_n, b, q)$ has empty singular continuous spectrum and no strictly positive eigenvalues, see, for instance, Erdogan, Goldberg, and Schlag [29], [30], Ikebe and Saitō [50], (see also [4], [10], [31], [35], [55], [89]); in particular, the analog of (5.3) holds for $\widetilde{A}_2(I_n, b, q)$.

Next, we recall the unperturbed operator $H_0 := -\Delta$, $\mathrm{dom}(H_0) = W^{2,2}(\mathbb{R}^n)$, and introduce the first-order perturbation term,

$$
L_1 f = 2i \sum_{k=1}^{n} b_k \partial_k f + (i\,\mathrm{div}(b) + |b|^2 + q)f, \quad f \in \mathrm{dom}(L_1) = W^{1,2}(\mathbb{R}^n). \tag{5.22}
$$

We denote the distorted plane waves associated with $\widetilde{A}_2(I_n, b, q)$ by $\phi(\,\cdot\,,\,\cdot\,)$, and abbreviate

$$
\phi_0(x,\xi) := e^{i\xi\cdot x}, \quad (x,\xi) \in \mathbb{R}^{2n}. \tag{5.23}
$$

In the following we will show that

$$
\sup_{\xi\in\mathbb{R}^n} \|\phi(\,\cdot\,,\xi)\|_{L^2(\Omega)} < \infty. \tag{5.24}
$$

To this end, we employ [30, Theorem 1.2] (see also [29, Theorem 2]) with $\alpha = 0$, $\sigma = 1$ and infer

$$
K := \sup_{|\xi|\geqslant 0} \left(\langle|\xi|\rangle \big\| \langle\cdot\rangle^{-2} \big(\widetilde{A}_2(I_n, b, q) - (|\xi|^2 \pm i0) \big)^{-1} \langle\cdot\rangle^{-2} \big\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \right) < \infty, \tag{5.25}
$$

abbreviating $\langle\cdot\rangle := \left[1 + (\,\cdot\,)^2\right]^{1/2}$.

The distorted plane wave $\phi(\,\cdot\,,\,\cdot\,)$ can again be chosen as one of $\phi_+(\,\cdot\,,\,\cdot\,)$ or $\phi_-(\,\cdot\,,\,\cdot\,)$ and be decomposed in the form

$$
\phi_\pm(x,\xi) = \phi_0(x,\xi) + \psi_\pm(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2n}, \tag{5.26}
$$

where

$$\psi_\pm(x,\xi) := -\left(\left(\widetilde{A}_2(I_n,b,q) - (|\xi|^2 \pm i0)\right)^{-1}(L_1\phi_0)\right)(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2n}. \quad (5.27)$$

(In this context we recall that

$$\begin{aligned}
|\xi|^2\phi_\pm(x,\xi) &= \left(\widetilde{A}_2(I_n,b,q)\phi_\pm\right)(x,\xi) \\
&= |\xi|^2\phi_0(x,\xi) + (L_1\phi_0)(x,\xi) + \left(\widetilde{A}_2(I_n,b,q)\psi_\pm\right)(x,\xi),
\end{aligned} \quad (5.28)$$

or equivalently,

$$-(L_1\phi_0)(x,\xi) = \left(\left(\widetilde{A}_2(I_n,b,q)\right) - |\xi|^2\right)\psi_\pm\right)(x,\xi), \quad (5.29)$$

in the sense of distributions, illustrating (5.27).)

One then infers

$$\begin{aligned}
&\|\psi_\pm(\,\cdot\,,\xi)\|_{L^2(\Omega)} \\
&\quad = \left\|\chi_\Omega\langle\,\cdot\,\rangle^2\langle\,\cdot\,\rangle^{-2}\left(\widetilde{A}_2(I_n,b,q) - (|\xi|^2 \pm i0)\right)^{-1}\langle\,\cdot\,\rangle^{-2}\langle\,\cdot\,\rangle^2(L_1\phi_0)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant \left\|\chi_\Omega\langle\,\cdot\,\rangle^2\right\|_{L^\infty(\mathbb{R}^n)}\left\|\langle\,\cdot\,\rangle^{-2}\left(\widetilde{A}_2(I_n,b,q) - (|\xi|^2 \pm i0)\right)^{-1}\langle\,\cdot\,\rangle^{-2}\right\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \\
&\qquad \times \left\|\langle\,\cdot\,\rangle^2(L_1\phi_0)\right\|_{L^2(\mathbb{R}^n)}. \quad (5.30)
\end{aligned}$$

Employing (5.25), the fact that Ω is bounded, and that the coefficients of L_1 have compact support (cf. (5.22)), one concludes

$$\begin{aligned}
&\left\|\langle\,\cdot\,\rangle^{-2}\left(\widetilde{A}_2(I_n,b,q) - (|\xi|^2 \pm i0)\right)^{-1}\langle\,\cdot\,\rangle^{-2}\right\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\left\|\langle\,\cdot\,\rangle^2(L_1\phi_0)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant K\langle|\xi|\rangle^{-1}\left\|\langle\,\cdot\,\rangle^2\phi_0(\,\cdot\,,\xi)\left(-2b\cdot\xi + i\operatorname{div}(b) + |b|^2 + q\right)\right\|_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant 2K|\xi|\langle|\xi|\rangle^{-1}\left\|\langle\,\cdot\,\rangle^2 b\right\|_{[L^2(\mathbb{R}^n)]^n} + K\langle|\xi|\rangle^{-1}\left\|\langle\,\cdot\,\rangle^2\left(i\operatorname{div}(b) + |b|^2 + q\right)\right\|_{L^2(\mathbb{R}^n)} \\
&\underset{|\xi|\to\infty}{=} O(1). \quad (5.31)
\end{aligned}$$

Combining (5.30) and (5.31) one obtains the required estimate (5.24). $\square$

APPENDIX A. THE MINIMIZATION OF $\mathcal{I}_K(\alpha) = \alpha^{-1}\int_{\mathbb{R}^n}\left[\alpha - |\eta|^{4m} + |\eta|^{2m}\right]_+ d^n\eta$ IN EQUATION (4.29)

In this appendix we carry out the explicit minimization in α for $\alpha > 0$ of the integral

$$\mathcal{I}_K(\alpha) := \alpha^{-1}\int_{\mathbb{R}^n}\left[\alpha - |\eta|^{4m} + |\eta|^{2m}\right]_+ d^n\eta. \quad (A.1)$$

Since the integral is only over the region of n-space where $\alpha - |\eta|^{4m} + |\eta|^{2m}$ is positive, and this function is radial, our problem immediately reduces to the minimization of α^{-1} times a radial integral in $r = |\eta|$. Since the function $r^{4m} - r^{2m} = r^{2m}(r^{2m} - 1)$ is negative on $0 < r < 1$ and is positive and increasing for $r > 1$, for $\alpha > 0$ the relation $\alpha = r^{4m} - r^{2m}$ implicitly determines a unique value $r_\alpha > 1$, with r_α^{2m} given explicitly by

$$r_\alpha^{2m} = \frac{1}{2} + \left(\alpha + \frac{1}{4}\right)^{1/2}. \quad (A.2)$$

It is clear that the value of r_α is a strictly increasing function of α and runs from 1 to ∞ as α runs from 0 to ∞.

By the reductions mentioned above, one obtains

$$\mathcal{I}_K(\alpha) = nv_n\alpha^{-1}\int_0^{r_\alpha}\left[\alpha + r^{2m} - r^{4m}\right]r^{n-1}\,dr, \quad (A.3)$$

where v_n is the volume of the ball of unit radius in $\mathbb{R}^n$ as mentioned with (4.13). Since the v_n here is included explicitly in (4.13), to prove (4.13), in what remains we will show that the function $f_{n,m}(\alpha)$ defined by

$$f_{n,m}(\alpha) := n\alpha^{-1} \int_0^{r_\alpha} [\alpha + r^{2m} - r^{4m}] \, r^{n-1} \, dr \qquad \text{(A.4)}$$

has minimum given by

$$\widetilde{f}_{n,m} := \left(1 + \frac{2m}{n + 2m}\right)^{n/(2m)}, \quad m, n \in \mathbb{N}. \qquad \text{(A.5)}$$

By integrating (A.4), it is easy to see that

$$f_{n,m}(\alpha) := n\alpha^{-1} \left[\frac{\alpha r_\alpha^n}{n} + \frac{r_\alpha^{n+2m}}{n + 2m} - \frac{r_\alpha^{n+4m}}{n + 4m}\right]. \qquad \text{(A.6)}$$

Replacing the explicit α appearing inside the square brackets here using $\alpha = r_\alpha^{4m} - r_\alpha^{2m}$ and simplifying, one finds

$$\alpha \, f_{n,m}(\alpha) = \frac{4m \, r_\alpha^{n+4m}}{n + 4m} - \frac{2m \, r_\alpha^{n+2m}}{n + 2m}. \qquad \text{(A.7)}$$

We shall have need of this expression shortly.

Next, some further properties of $f_{n,m}$ and its derivative will be developed. One has

$$\alpha \, f_{n,m}(\alpha) = n \int_0^{r_\alpha} [\alpha + r^{2m} - r^{4m}] \, r^{n-1} \, dr, \qquad \text{(A.8)}$$

and therefore, by Leibniz's rule,

$$[\alpha \, f_{n,m}(\alpha)]' = n[\alpha + r_\alpha^{2m} - r_\alpha^{4m}] \, r_\alpha^{n-1} \, r_\alpha' + n \int_0^{r_\alpha} r^{n-1} \, dr = r_\alpha^n, \qquad \text{(A.9)}$$

with the simplification in the last step occurring due to the implicit relation defining r_α. From (A.9) it follows that

$$\alpha \, f'_{n,m}(\alpha) = r_\alpha^n - f_{n,m}(\alpha), \qquad \text{(A.10)}$$

and hence, using (A.7) and $\alpha = r_\alpha^{4m} - r_\alpha^{2m}$, that

$$\begin{aligned}
\alpha^2 f'_{n,m}(\alpha) &= \alpha \, r_\alpha^n - \left[\frac{4m \, r_\alpha^{n+4m}}{n + 4m} - \frac{2m \, r_\alpha^{n+2m}}{n + 2m}\right] \\
&= n\left(r_\alpha^{2m} - \frac{n + 4m}{n + 2m}\right) \frac{r_\alpha^{n+2m}}{n + 4m}.
\end{aligned} \qquad \text{(A.11)}$$

It is now clear that $f_{n,m}(\alpha)$ has a minimum on $\alpha \in (0, \infty)$, and that it occurs at

$$r_\alpha^{2m} = \frac{n + 4m}{n + 2m} := \widetilde{r}_\alpha^{2m} \qquad \text{(A.12)}$$

(one notes that this value is clearly larger than 1, and hence corresponds to an $\alpha > 0$). The corresponding value of α, denoted by $\widetilde{\alpha}$, may then be computed as

$$\widetilde{\alpha} = \widetilde{r}_\alpha^{2m}(\widetilde{r}_\alpha^{2m} - 1) = \frac{n + 4m}{n + 2m} \frac{2m}{n + 2m} = \frac{2m(n + 4m)}{(n + 2m)^2}. \qquad \text{(A.13)}$$

Finally one computes $\widetilde{f}_{n,m}$ using (A.7), (A.12), and (A.13), which leads to

$$\widetilde{f}_{n,m} = f_{n,m}(\widetilde{\alpha}) = \left(\frac{n + 4m}{n + 2m}\right)^{n/(2m)} = \left(1 + \frac{2m}{n + 2m}\right)^{n/(2m)}, \quad m, n \in \mathbb{N}, \quad \text{(A.14)}$$

in accordance with our statement above. This completes the proof of Theorem 4.3.

We conclude with some remarks comparing the constant $\widetilde{f}_{n,m}$ found here with the corresponding constants $\widetilde{g}_{n,m}$ (our notation) found by Laptev in [61] (the comparison is most apt if we restrict our attention to the case of the Laplacian (i.e., $a = I_n$, $b = q = 0$), as that is the main case considered by Laptev [61]). Laptev's $\widetilde{g}_{n,m}$ are given by

$$\widetilde{g}_{n,m} = \left(1 + \frac{2m}{n}\right)^{n/(2m)}, \quad m, n \in \mathbb{N}. \tag{A.15}$$

It is clear from these expressions that

$$\widetilde{f}_{n,m} < \widetilde{g}_{n,m}, \quad m, n \in \mathbb{N}. \tag{A.16}$$

This shows that the bound given in Theorem 4.3 is always better than the bound (2.24) combined with the earlier work of Laptev [61]. Of course in the large n limit (for fixed m) both constants become arbitrarily close, since the limit of either $\widetilde{g}_{n,m}$ or $\widetilde{f}_{n,m}$ as $n \to \infty$ is $e \approx 2.71828$. On the other hand, in the large m limit (with n fixed) both constants go to 1 from above (with 1 being the best possible value of the constant that could obtain in our upper bounds, at least in the case of the Laplacian).

In fact, it is generally true that

$$1 < \widetilde{f}_{n,m} < \widetilde{g}_{n,m} < e, \quad m, n \in \mathbb{N}, \tag{A.17}$$

that is, that

$$1 < (1 + 2m/(n+2m))^{n/2m} < (1 + (2m/n))^{n/(2m)} < e, \quad m, n \in \mathbb{N}, \tag{A.18}$$

with 1 and e being the best possible lower and upper bounds for both $\widetilde{f}_{n,m}$ and $\widetilde{g}_{n,m}$ for all $m, n \in \mathbb{N}$. These claims can be proved using elementary calculus by focusing on the functions $G(x) := (\ln(1+x))/x$ and $F(x) := (\ln(1 + x/(1+x)))/x$ for $x > 0$ (note that with the identification $x = 2m/n$ these are the logarithms of $\widetilde{g}_{n,m}$ and $\widetilde{f}_{n,m}$, respectively, and that all $x > 0$ can be approximated arbitrarily closely by such ratios for $m, n \in \mathbb{N}$). In fact, one can show that the functions $G(x)$ and $F(x)$ are both strictly decreasing on $(0, \infty)$, with limiting value 1 as $x \to 0^+$, and with limiting value 0 as $x \to \infty$. This implies, in particular, that in all upper bound formulas for counting functions $N(\cdot)$ in this paper the bound would continue to hold (as a strict inequality) if the constant represented by $(1+2m/(n+2m))^{n/2m}$ were replaced by the value e.

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