EIGENVALUE BOUNDS OF MIXED STEKLOV PROBLEMS

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(with an appendix by F. Ferrulli and J. Lagacé)

Abstract. We study bounds on the Riesz means of the mixed Steklov–Neumann and Steklov–Dirichlet eigenvalue problem on a bounded domain Ω in $\mathbb{R}^n$. The Steklov–Neumann eigenvalue problem is also called the sloshing problem. We obtain two-term asymptotically sharp lower bounds on the Riesz means of the sloshing problem and also provide an asymptotically sharp upper bound for the Riesz means of mixed Steklov–Dirichlet problem. The proof of our results for the sloshing problem uses the average variational principle and monotonicity of sloshing eigenvalues. In the case of Steklov-Dirichlet eigenvalue problem, the proof is based on a well–known bound on the Riesz means of the Dirichlet fractional Laplacian, and an inequality between the Dirichlet and Navier fractional Laplacian. The two-term asymptotic results for the Riesz means of mixed Steklov eigenvalue problems are discussed in the appendix which in particular show the asymptotic sharpness of the bounds we obtain.

1. Introduction

1.1. **Statement of the problem.** Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz and piecewise smooth boundary $\partial\Omega$. We assume that

$$\partial\Omega = \mathcal{F} \cup \mathcal{B}, \quad \text{where } \mathcal{F} \subset \{x_n = 0\}, \text{ and } \mathcal{B} \subset \{x_n < 0\}. \tag{1.1}$$

Throughout the paper, we refer to $\mathcal{F}$ as a subset of $\mathbb{R}^{n-1} \times \{0\}$ and as a subset of $\mathbb{R}^{n-1}$ interchangeably. Consider the following eigenvalue problem

$$\begin{cases} \Delta f = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}} f = 0, & \text{on } \mathcal{B}, \\ \partial_{\mathbf{e_n}} f = \nu f, & \text{on } \mathcal{F}, \end{cases} \tag{1.2}$$

where $\partial_{\mathbf{n}} f$ and $\partial_{\mathbf{e_n}} f$ are the derivative of f in the direction of the unit outward normal vector along $\mathcal{B}$ and $\mathcal{F}$ denoted by $\mathbf{n}$ and $\mathbf{e_n}$ respectively. The above mixed Steklov–Neumann eigenvalue problem is also called the sloshing problem. It is known that it has a discrete set of eigenvalues (see for example [2, Chapter III])

$$0 = \nu_1 \le \nu_2 \le \nu_3 \le \cdots \nearrow \infty$$

and each eigenvalue has a finite multiplicity. The corresponding eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$ restricted to the free surface $\mathcal{F}$ form a basis for $L^2(\mathcal{F})$. The eigenvalues of the sloshing problem can be considered as the eigenvalues of the Dirichlet–to–Neumann map

$$\mathcal{D}_N : L^2(\mathcal{F}) \to L^2(\mathcal{F}),$$
$$f \mapsto \partial_{\mathbf{e_n}} \tilde{f},$$

where $\tilde{f}$ is the harmonic extension of f to Ω satisfying the Neumann boundary condition on $\mathcal{B}$. The sloshing problem naturally appears in the study of the sloshing liquid, where the sloshing frequency is proportional to $\sqrt{\nu_j}$ and its study has a long history. For a short historical note we refer to [11], and for more recent developments on the subject to [1, 16] and the references therein.

The focus of our study is to find sharp semiclassical lower/upper bounds for the Riesz means of eigenvalues of the mixed Steklov problems (1.2) and (1.5) (see below).
The Riesz mean $R_\gamma(z)$ of order $\gamma > 0$ is defined as

$$R_\gamma(z) := \sum_j (z-\nu_j)_+^\gamma, \qquad z>0,$$

where $(z-\nu)_+ := \max\{0, z-\nu\}$. We may also denote it by $R_\gamma^\Omega(z,\mathcal{D}_N)$ to identify the domain and the operator under consideration.
When $\gamma \to 0$, it approaches the counting function

$$N(z) := \sum_{\nu_j<z} 1 = \sup\{k : \nu_k < z\}$$

and by convention we denote $R_0(z) := N(z)$. The asymptotics of the counting function $N(z)$ for the eigenvalue problem (1.2) is given by (see for example [22])

$$N(z) \sim \frac{\omega_{n-1}}{(2\pi)^{n-1}} |\mathcal{F}| z^{n-1}, \qquad z \nearrow \infty,$$

where $\omega_{n-1} = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}$ is the volume of a unit ball in $\mathbb{R}^{n-1}$, and $|\mathcal{F}|$ denote the $(n-1)$-Euclidean volume of $\mathcal{F}$. Using the Riesz iteration, i.e. the following identities

$$R_{\gamma+\rho}(z) = \frac{\Gamma(\gamma+\rho+1)}{\Gamma(\gamma+1)\ \Gamma(\rho)} \int_0^\infty (z-t)_+^{\rho-1} R_\gamma(t)dt \tag{1.3}$$

and

$$R_\gamma(z) = \gamma \int_0^\infty (z-t)_+^{\gamma-1} R_0(t)dt = \gamma \int_0^z (z-t)^{\gamma-1} R_0(t)dt,$$

we can immediately get the asymptotics behaviour of $R_\gamma(z)$

$$R_\gamma(z) \sim C_{n,\gamma} |\mathcal{F}| z^{n+\gamma-1}, \qquad z \nearrow \infty, \tag{1.4}$$

where

$$C_{n,\gamma} := \frac{1}{(4\pi)^{\frac{n-1}{2}}} \frac{\Gamma(\gamma+1)\Gamma(n)}{\Gamma(\frac{n+1}{2})\Gamma(n+\gamma)}.$$

In [9], Ivrii studied the two-term asymptotic expansion for the Steklov eigenvalues. This suggests that for mixed Steklov problems one can expect that (1.4) can be improved to

$$R_\gamma(z) = C_{n,\gamma}|\mathcal{F}| z^{n+\gamma-1} + O(z^{n+\gamma-2}), \qquad z \nearrow \infty.$$

For basic facts on the Riesz means, we refer to [3, 7]. Sharp semiclassical bounds on the Riesz means of Dirichlet and Neumann eigenvalues of the Laplacian were studied in numerous work, see for example [13, 6, 8, 17]. Recently, a sharp semiclassical bound for the Riesz means $R_\gamma(z)$, $\gamma \geq 2$, of Steklov eigenvalues was obtained in [21]. However, such sharp semiclassical bounds for Riesz means of the mixed Steklov problem are unknown.
We also consider the mixed Steklov–Dirichlet eigenvalue problem:

$$\begin{cases} \Delta f = 0, & \text{in } \Omega, \\ f = 0, & \text{on } \mathcal{B}, \\ \partial_{\mathbf{e_n}} f = \eta f, & \text{on } \mathcal{F}, \end{cases} \tag{1.5}$$

where instead of the Neumann boundary condition, the Dirichlet boundary condition is imposed on $\mathcal{B}$. For a physical interpretation of this problem, see for example [16]. The eigenvalues of

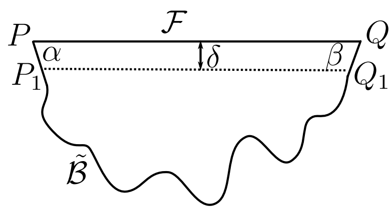

FIGURE 1.

the Steklov–Dirichlet problem can be also considered as the the eigenvalues of the Dirichlet–to–Neumann map

$$\mathcal{D}_D : L^2(\mathcal{F}) \to L^2(\mathcal{F})$$
$$f \mapsto \partial_{\mathbf{e_n}} \tilde{f},$$

where $\tilde{f}$ is the harmonic extension of f to Ω satisfying the Dirichlet boundary condition on $\mathcal{B}$. We denote the Riesz means of eigenvalues of problem (1.5) by $R_\gamma^\Omega(z, \mathcal{D}_D)$, and when there is no confusion by $R_\gamma^\Omega(z)$ or $R_\gamma(z)$.
In the following subsections, we state our main results on asymptotically sharp bounds for $R_\gamma^\Omega(z, \mathcal{D}_N)$ and $R_\gamma^\Omega(z, \mathcal{D}_D)$. As a consequence, we get asymptotically sharp bounds on the sum of first k eigenvalues of the mixed Steklov problem.

1.2. **Steklov–Neumann eigenvalue problem.** Our first result concerns a two–term asymptotically sharp lower bound on $R_\gamma(z, \mathcal{D}_N)$ in dimension two.

Theorem 1.1. *Let $\tilde{\Omega}$ be a bounded domain in $\mathbb{R}^2$ with $\partial\tilde{\Omega} = \mathcal{F} \cup \tilde{\mathcal{B}}$ as in (1.1). We assume that $\mathcal{F}$ is connected and there exists $\delta > 0$ such that $\tilde{\mathcal{B}}$ meets $\mathcal{F}$ in two line segments in a $\delta-$neighbourhood of corner points[1] P and Q (as shown in Figure 1) with angles $\alpha, \beta \in (0, \pi)$. We denote the complement of these two line segments in $\tilde{\mathcal{B}}$ by $\tilde{\mathcal{B}}_c$. Then for every $\gamma \geq 1$ and every $z > 0$ there exists a constant $c = c(z, \gamma, \delta, \alpha, \beta, |\tilde{\mathcal{B}}_c|)$ depending on $z, \gamma, \delta, \alpha, \beta$, and $|\tilde{\mathcal{B}}_c|$ such that for any $\Omega \subset \tilde{\Omega}$ with $\partial\Omega = \mathcal{F} \cup \mathcal{B}$ the Riesz mean $R_\gamma^\Omega(z, \mathcal{D}_N)$ satisfies the following inequality.*

$$R_\gamma^\Omega(z, \mathcal{D}_N) \geq C_{2,\gamma}|\mathcal{F}|z^{\gamma+1} + \frac{1}{2\pi}\left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right) z^\gamma + c, \tag{1.6}$$

where $C_{2,\gamma} = \frac{1}{\pi(\gamma+1)}$. Moreover, $c = O(z^{\gamma-1})$ as $z \to \infty$. Here our convention is that $\frac{1}{\tan(\pi/2)} = 0$.

Remark 1.2. Note that for a fixed z, when δ tends to 0, constant c in inequality (1.6) may tend to $-\infty$. However, there exists a constant z_0 depending on $\gamma, \delta, \alpha, \beta$, and $|\tilde{\mathcal{B}}_c|$ with $(\alpha, \beta) \neq (\frac{\pi}{2}, \frac{\pi}{2})$ such that for every $z \geq z_0$ and $\gamma \geq 1$, and for any domain Ω satisfying the assumption of Theorem 1.1, we have,

$$R_\gamma^\Omega(z, \mathcal{D}_N) \geq C_{2,\gamma}|\mathcal{F}|z^{\gamma+1} + \frac{1}{4\pi}\left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right) z^\gamma.$$

When $\tilde{\Omega}$ is a triangle/trapezoid, then for δ equal to the height of the triangle/trapezoid, we have $\tilde{\mathcal{B}}_c = \emptyset$, and constant c in (1.6) is equal to

$$c = \frac{1}{4\pi\delta}\left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right)(1 - e^{-2\delta z}).$$

[1]The points in the intersection of $\bar{\mathcal{F}}$ and $\bar{\mathcal{B}}$ are called the corner points.

The coefficient of z^γ in (1.6) is zero when $(\alpha,\beta)=(\frac{\pi}{2},\frac{\pi}{2})$. When Ω is a subset of the infinite strip $\mathcal{F}\times(-\infty,0)$, we say that Ω satisfies the so-called *John condition*, see [1]. We show that when Ω satisfies the John condition then we get a uniform lower bound with $\frac{1}{2}z^\gamma$ in the second term:

$$R_\gamma^\Omega(z,\mathcal{D}_N)\geq \frac{|\mathcal{F}|}{\pi(\gamma+1)}z^{\gamma+1}+\frac{1}{2}z^\gamma. \tag{1.7}$$

We remark that inequality (1.7) is not a consequence of Theorem 1.1, see Proposition 2.8.

In Theorem 1.1, when $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are tangent to each other at points P and Q, then the coefficient of z^γ depends only on the interior angles between $\mathcal{F}$ and $\mathcal{B}$.

Recently, Levitin, Parnovski, Polterovich and Sher in [16, Theorem 1.2.2] showed that when $\alpha,\beta\in(0,\frac{\pi}{2})$ are the interior angles between $\mathcal{F}$ and $\mathcal{B}$, and $k\nearrow\infty$, the following asymptotic expansion holds

$$\nu_k|\mathcal{F}|=\pi k-\frac{\pi}{2}-\frac{\pi^2}{8}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+o(1). \tag{1.8}$$

Their result in particular proves Fox-Kuttler's conjecture in 1983 [5]. From (1.8), one can deduce $N(z)=\frac{1}{\pi}|\mathcal{F}|z+O(1)$, as $z\nearrow\infty$ (see also [16, Corollary 1.6.1] for a related result). Ferrulli and Lagacé show (see the appendix) that the following two-term asymptotic for $R_\gamma(z)$, $\gamma>0$ is a consequence of (1.8).

$$R_\gamma(z,\mathcal{D}_N)=C_{2,\gamma}|\mathcal{F}|z^{\gamma+1}+\frac{\pi}{8}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)z^\gamma+o(z^\gamma), \tag{1.9}$$

where $\alpha,\beta\in(0,\frac{\pi}{2})$ are the interior angles between $\mathcal{F}$ and $\mathcal{B}$. The above asymptotic holds when one or both angles take the value $\frac{\pi}{2}$ provided that Ω satisfies the *local John condition* (we refer to the appendix for the definition). We observe that the coefficient of z^γ in the second term of inequality (1.6) depends on the same quantities appearing in the coefficient of z^γ in the two–term asymptotic expansion (1.8). In particular, for domains satisfying the John condition, the second term of (1.7) is also asymptotically sharp.

In higher dimensions, we obtain a general formula for a two–term lower bound on $R_\gamma(z,\mathcal{D}_N)$, $\gamma\geq 1$, see Theorem 2.3. Here, we only mention a corollary of Theorem 2.3.

Theorem 1.3. *Assume that Ω with $\partial\Omega=\mathcal{F}\cup\mathcal{B}$ satisfied the John condition, i.e. is a subset of $\mathcal{F}\times(-\infty,0)$. Then*

$$R_1^\Omega(z)\geq C_{n,1}|\mathcal{F}|z^n+\frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}}\frac{|\mathcal{F}|}{(2h_\Omega)^n}\left(\Gamma(n)-\Gamma(n,2h_\Omega z)\right), \tag{1.10}$$

where h_Ω is the depth of Ω.

We recall the definition of the incomplete Γ-function $\Gamma(n,x)$:

$$\Gamma(n,x):=(n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}.$$

In particular, notice

$$\Gamma(n)-\Gamma(n,x)>0,\qquad\forall x>0,\quad\forall n\in\mathbb{N}.$$

One can apply the Riesz iteration in the inequalities above and get a lower bound on $R_\gamma^\Omega(z,\mathcal{D}_N)$. The leading term in (1.3) is asymptotically sharp. The proofs of the results above are based on using the averaged variational principal introduced in [4] and monotonicity results for eigenvalues of (1.2) studied in [1].

We now give an asymptotically sharp bounds for the sum of first k eigenvalues of the sloshing problem. Kröger [10] obtained a sharp upper bound for the sum of eigenvalues of the Laplacian with Neumann boundary condition. His result was recently sharpen by Harrell and Stubbe in [8]. Adapting the argument in [8] to the sloshing eigenvalue problem, we obtain a counterpart of the Kröger–Harrell–Stubbe inequality for the sum of eigenvalues of the sloshing problem.

Theorem 1.4. *Under the assumption of Theorem 1.3, we have*

$$\frac{1}{k}\sum_{j=1}^{k}\nu_j \leq \frac{n-1}{n}\left(W_{n,k} - \frac{1}{W_{n,k}}(\nu_{k+1}-W_{n,k})^2\right), \tag{1.11}$$

where

$$W_{n,k} := 2\pi\omega_{n-1}^{-\frac{1}{n-1}}\left(\frac{k}{|\mathcal{F}|}\right)^{\frac{1}{n-1}}.$$

Inequality (1.11) *in particular gives a two-sided asymptotically sharp bound for an individual eigenvalue*

$$W_{n,k}(1-\sqrt{1-S_k}) \leq \nu_{k+1} \leq W_{n,k}(1+\sqrt{1-S_k}), \tag{1.12}$$

where

$$S_k := \frac{n}{n-1}\frac{\sum_{j=1}^{k}\nu_j}{kW_{n,k}}.$$

1.3. **Steklov–Dirichlet eigenvalue problem.** In this section, we state our result on asymptoticly sharp upper bounds on $R_\gamma^\Omega(z,\mathcal{D}_D)$, $\gamma \geq 1$. The same asymptotic (1.4) remains true for the mixed Steklov–Dirichlet problem (1.5).

$$R_\gamma(z) \sim C_{n,\gamma}|\mathcal{F}|z^{n+\gamma-1}, \qquad z \nearrow \infty. \tag{1.13}$$

We obtain an asymptotically sharp upper bound for $R_\gamma^\Omega(z,\mathcal{D}_D)$:

Theorem 1.5. *Let Ω be a bounded domain and subset of an infinite cylinder $\mathcal{F}\times[-\infty,0]$, where $\partial\Omega = \mathcal{F}\cup\mathcal{B}$. Here $\mathcal{F}$ is the free part of the boundary. Then for every $\gamma \geq 1$ and $z > 0$ we have*

$$R_\gamma^\Omega(z,\mathcal{D}_D) = \sum_j (z-\eta_j)_+^\gamma \leq C_{n,\gamma}|\mathcal{F}|z^{n+\gamma-1}. \tag{1.14}$$

For the proof of this theorem, we use the relationship between eigenvalues of the fractional Laplacian with different type of boundary conditions studied in [18, 19, 20] together with the result of Laptev [13] on upper bounds of the Riesz means for the Dirichlet fractional Laplacian. As pointed out in [15] and [14, Page 8], an asymptotically sharp upper bound for $R_1(z)$ leads to an asymptotically sharp lower bound for the sum of first k eigenvalues (and vice versa) by using the Legendre transform. Thus, we get the following bound on the sum of first k Steklov-Dirichlet eigenvalues by applying the Legendre transform to (1.14),:

Corollary 1.6. *Under the assumption of Theorem 1.5, the following inequality holds.*

$$\frac{1}{k}\sum_{j=1}^{k}\eta_j \geq \frac{n-1}{n}W_{n,k} = 2\pi\left(\frac{n-1}{n}\right)\omega_{n-1}^{-\frac{1}{n-1}}\left(\frac{k}{|\mathcal{F}|}\right)^{\frac{1}{n-1}}. \tag{1.15}$$

As in (1.9), Ferrulli and Lagacé also obtain the following two-term asymptotic for $R_\gamma(z, \mathcal{D}_N)$.

$$R_\gamma(z, \mathcal{D}_D) = C_{2,\gamma}|\mathcal{F}|z^{\gamma+1} - \frac{\pi}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) z^\gamma + o(z^\gamma), \tag{1.16}$$

where $\alpha, \beta \in (0, \frac{\pi}{2})$ are the interior angles between $\mathcal{F}$ and $\mathcal{B}$. This asymptotic holds when α or β is equal to $\frac{\pi}{2}$ provided that Ω satisfies the local John condition. We refer to the appendix for more details.
We show that when Ω satisfies the John condition, we can get two-term asymptotically sharp upper bound for $R_1^\Omega(z, \mathcal{D}_D)$.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^2$ satisfies the John condition. Then the Riesz mean of eigenvalues η_j of problem* (1.5) *satisfies*

$$R_1^\Omega(z, \mathcal{D}_D) = \sum_j (z - \eta_j)_+ \leq \frac{|\mathcal{F}|}{2\pi} z^2 - \frac{1}{2}z + \frac{\pi}{2|\mathcal{F}|}. \tag{1.17}$$

One can apply the Riesz iteration to find bounds on $R_\gamma(z, \mathcal{D}_D)$.

Open Question 1.8. *It is an intriguing question if we can get a two–term upper bound with a negative second term depending only on α and β. One can ask if there exist a positive constant $C(\alpha, \beta)$ such that*

$$\sum_j (z - \eta_j)_+^\gamma \leq C_{n,\gamma}|\mathcal{F}|z^{n+\gamma-1} - C(\alpha, \beta)z^{n+\gamma-2}.$$

The paper is organised as follows. In section 2, we prove the main results on bounds on the Riesz means of eigenvalues of problem (1.2), and on Kröger-Harrell-Stubbe's type inequality for eigenvalues of the sloshing problem. We also consider cases in which we can get more explicit lower bounds. In section 3, we study the upper bounds on the Riesz means of eigenvalues of the mixed Steklov-Dirichlet problem (1.5).

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2. Sloshing eigenvalue problem

In this section, we prove the results of Section 1.2 of the introduction. We first recall the variational characterisation of the eigenvalues of the mixed Steklov-Neumann problem. Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of eigenfunctions associate with $\{\nu_j\}_{j=1}^\infty$. The k-th eigenvalue ν_k of problem (1.2) can be characterised by

$$\nu_k = \inf_{0 \neq f \in H_k} \mathfrak{R}(f), \tag{2.1}$$

where $H_k = \left\{g \in H^1(\Omega) : \int_\mathcal{F} g\varphi_j ds = 0, j = 1, \ldots, k-1\right\}$, and

$$\mathfrak{R}(f) := \frac{\int_\Omega |\nabla f|^2 dx}{\int_\mathcal{F} |f|^2 ds}.$$

Let $\mathcal{H}(\Omega)$ denotes the space of harmonic functions on Ω. In (2.1) one can replace H_k by $\mathcal{H}_k := \{f \in \mathcal{H}(\Omega) : \int_\mathcal{F} g\varphi_j ds = 0, j = 1, \ldots, k-1\}$. We recall the so-called averaged variational principle introduced in [4]. Let $f \in H^1(\Omega)$ and $z \in (\nu_{k-1}, \nu_k]$. We choose $\{\varphi_j\}$ so that their restriction to $\mathcal{F}$ forms an orthonormal basis for $L^2(\mathcal{F})$. Thus, by (2.1) we have

$$z \leq \Re\left(f - \sum_{j=1}^{k-1} \langle \varphi_j, f\rangle_{\mathcal{F}} \varphi_j\right) = \frac{\int_\Omega |\nabla f|^2\, dx - \sum_{j=1}^{k-1} \nu_j |\langle \varphi_j, f\rangle|^2_{\mathcal{F}}}{\int_{\mathcal{F}} |f|^2\, ds - \sum_{j=1}^{k-1} |\langle \varphi_j, f\rangle|^2_{\mathcal{F}}},$$

where $\langle f, g\rangle_{\mathcal{F}} := \int_{\mathcal{F}} f\bar{g}\, ds$. Therefore,

$$\sum_j (z - \nu_j)_+ |\langle \varphi_j, f\rangle|^2_{\mathcal{F}} \geq z \int_{\mathcal{F}} |f|^2 ds - \int_\Omega |\nabla f|^2 dx. \tag{2.2}$$

If moreover $f \in \mathcal{H}(\Omega)$, then applying the Green formula we get

$$\sum_j (z - \nu_j)_+ |\langle \varphi_j, f\rangle|^2_{\mathcal{F}} \geq z \int_{\mathcal{F}} |f|^2 ds - \mathrm{Re} \int_{\partial\Omega} \frac{\partial f}{\partial \mathbf{n}} \bar{f}\, ds. \tag{2.3}$$

We summarise the discussion above in the following lemma which is called the averaged variational principle. This is an special case of a more general statement in [4, Lemma 1.5].

Lemma 2.1 (averaged variational principle)**.** *Let $f_\xi \in \mathcal{H}(\Omega)$ be a family of harmonic functions where ξ varies over a measure space $(M, \mathcal{M}, \mu)$, with σ-algebra $\mathcal{M}$. Let M_0 be a measurable subset of M. Then for any $z \in \mathbb{R}_+$ we have*

$$\sum_j (z - \nu_j)_+ \int_M |\langle \varphi_j, f_\xi\rangle|^2_{\mathcal{F}} d\mu \geq z \int_{M_0} \int_{\mathcal{F}} |f_\xi|^2 ds\, d\mu - \int_{M_0} \mathrm{Re} \int_{\partial\Omega} \frac{\partial f_\xi}{\partial \mathbf{n}} \bar{f}_\xi\, ds\, d\mu. \tag{2.4}$$

Another key lemma we need is the monotonicity results for the mixed Steklov-Neumann eigenvalues:

Lemma 2.2. [1, Proposition 3.2.1] *Let Ω and $\tilde{\Omega}$ be subdomains of $\mathbb{R}^n$ whose boundaries $\partial\Omega = \mathcal{F} \cup \mathcal{B}$ and $\partial\tilde{\Omega} = \tilde{\mathcal{F}} \cup \tilde{\mathcal{B}}$ are as described in* (1.1)*. Let Ω be a proper subset of $\tilde{\Omega}$ and $\tilde{\mathcal{F}} = \mathcal{F}$. Then the following inequality holds.*

$$\nu_k(\Omega) < \nu_k(\tilde{\Omega}), \qquad \forall k \geq 2.$$

In particular,

$$R^\Omega_\gamma(z) = \sum_j (z - \nu_j(\Omega))_+ \geq \sum_j (z - \nu_j(\tilde{\Omega}))_+ = R^{\tilde{\Omega}}_\gamma(z).$$

We can now state a general form of the results mentioned in the introduction.

Theorem 2.3. *Let Ω be a bounded domain of $\mathbb{R}^n$ and $\partial\Omega = \mathcal{F} \cup \mathcal{B}$ as described in* (1.1)*. The Riesz means $R_\gamma(z)$, $\gamma \geq 1$, of the eigenvalues of the mixed Steklov-Neumann problem* (1.2) *satisfy the following inequality.*

$$R_\gamma(z) \geq C_{n,\gamma} |\mathcal{F}| z^{n+\gamma-1} + A_{n,\gamma}(z), \tag{2.5}$$

where

$$A_{n,\gamma}(z) = -\frac{(n-1)\gamma(\gamma-1)}{(4\pi)^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} \int_0^z (z-\eta)^{\gamma-2} \int_0^\eta \int_{\mathcal{B}} \langle \mathbf{n}(x), \mathbf{e_n}\rangle e^{2x_n r} r^{n-1}\, ds(x) dr d\eta.$$

Here $ds(x)$ is the volume element on $\mathcal{B}$ and $\langle \cdot, \cdot\rangle$ is the inner product in $\mathbb{R}^n$.

Note that

$$A_{n,1}(z) := -\frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \int_0^z \int_{\mathcal{B}} \langle \mathbf{n}(x), \mathbf{e_n}\rangle e^{2x_n r} r^{n-1}\, ds(x) dr. \tag{2.6}$$

It is clear that when $\langle \mathbf{n}(x), \mathbf{e_n}\rangle \leq 0$ for all $x \in \mathcal{B}$, then $A_{n,\gamma}(z)$ is positive. Below, we discuss situations in which we have more explicit estimates on $A_{n,1}$. Estimates on $A_{n,\gamma}$ follow by using the Riesz itteration.

Corollary 2.4. *Assume that there exists $\delta > 0$ such that*

$$\delta \leq \min\{|x_n| : x = (x', x_n) \in \mathcal{B},\ \langle \mathbf{n}(x), \mathbf{e_n}\rangle > 0\}.$$

Then

$$\begin{aligned} A_{n,1}(z) \geq \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \Bigg(\left(\int_{\mathcal{B}^-} |\langle \mathbf{n}(x), \mathbf{e_n}\rangle| ds \right) \frac{1}{(2h_\Omega)^n} \left(\Gamma(n) - \Gamma(n, 2h_\Omega z)\right) \\ - \left(\int_{\mathcal{B}^+} \langle \mathbf{n}(x), \mathbf{e_n}\rangle ds \right) \frac{1}{(2\delta)^n} \left(\Gamma(n) - \Gamma(n, 2\delta z)\right) \Bigg), \end{aligned} \tag{2.7}$$

where $\mathcal{B}^+ := \{x \in \mathcal{B} : \langle \mathbf{n}(x), \mathbf{e_n}\rangle > 0\}$, and $\mathcal{B}^- := \{x \in \mathcal{B} : \langle \mathbf{n}(x), \mathbf{e_n}\rangle \leq 0\}$. In particular, when $\mathcal{B}^+ = \emptyset$, we have

$$A_{n,1}(z) \geq \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \left(\left(\int_{\mathcal{B}^-} \langle \mathbf{n}(x), \mathbf{e_n}\rangle ds \right) \frac{1}{(2h_\Omega)^n} \left(\Gamma(n) - \Gamma(n, 2h_\Omega z)\right) \right). \tag{2.8}$$

Proof. By Theorem 2.3 we have

$$\begin{aligned} A_{n,1}(z) = \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \Bigg(\int_0^z \int_{\mathcal{B}^-} |\langle \mathbf{n}(x), \mathbf{e_n}\rangle| e^{2x_n r} r^{n-1}\, ds(x) dr \\ - \int_0^z \int_{\mathcal{B}^+} \langle \mathbf{n}(x), \mathbf{e_n}\rangle e^{2x_n r} r^{n-1}\, ds(x) dr \Bigg). \end{aligned}$$

Since $x_n < 0$, the function $e^{2x_n r}$ is decreasing. Therefore,

$$\begin{aligned} \int_0^z \int_{\mathcal{B}^-} |\langle \mathbf{n}(x), \mathbf{e_n}\rangle| e^{2x_n r} r^{n-1}\, ds(x) dr &\geq \left(\int_{\mathcal{B}^-} |\langle \mathbf{n}(x), \mathbf{e_n}\rangle| ds(x) \right) \left(\int_0^z e^{-2h_\Omega r} r^{n-1}\, dr \right) \\ &= \left(\int_{\mathcal{B}^-} |\langle \mathbf{n}(x), \mathbf{e_n}\rangle| ds(x) \right) \frac{1}{(2h_\Omega)^n} \left(\Gamma(n) - \Gamma(n, 2h_\Omega z)\right). \end{aligned}$$

Similarly

$$\begin{aligned} \int_0^z \int_{\mathcal{B}^+} \langle \mathbf{n}(x), \mathbf{e_n}\rangle e^{2x_n r} r^{n-1}\, ds(x) dr &\leq \left(\int_{\mathcal{B}^+} \langle \mathbf{n}(x), \mathbf{e_n}\rangle ds(x) \right) \left(\int_0^z e^{-2\delta r} r^{n-1}\, dr \right) \\ &= \left(\int_{\mathcal{B}^+} \langle \mathbf{n}(x), \mathbf{e_n}\rangle ds(x) \right) \frac{1}{(2\delta)^n} \left(\Gamma(n) - \Gamma(n, 2\delta z)\right). \end{aligned}$$

This completes the proof. □

Theorem 1.3 is an immediate consequence of Theorem 2.3 and Corollary 2.4.

Proof of Theorem 1.3. Let $\tilde{\Omega} := \mathcal{F} \times (-h_\Omega, 0)$. According to Lemma 2.2

$$R_1^\Omega(z) \geq R_1^{\tilde{\Omega}}(z).$$

Thus, it is enough to find a lower bound for $\tilde{R}_1(z)$. Using Corollary 2.4 we conclude

$$R_1^{\tilde{\Omega}}(z) \geq C_{n,1}|\mathcal{F}| z^n + \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \left(\frac{|\mathcal{F}|}{(2h_\Omega)^n} \left(\Gamma(n) - \Gamma(n, 2h_\Omega z)\right) \right).$$

□

Remark 2.5. For a cylindrical domain $\mathcal{F} \times (-h, 0)$, the sloshing eigenvalues can be calculated explicitly using separation of variable (see [1]). They are of the form

$$\sqrt{\mu_k}\tanh(\sqrt{\mu_k}h_\Omega),$$

where μ_k is the k-th Neumann eigenvalues of the Laplacian on $\mathcal{F}$. One can try to get an estimate for the Riesz means using this explicit expression of the eigenvalues. We shall see below that it does not give an asymptotically sharp bound.
For Ω we have

$$\begin{aligned} R_1(z) &= \sum_k (z - \sqrt{\mu_k}\tanh(\sqrt{\mu_k}h))_+ \\ &\geq \sum_k (z - \sqrt{\mu_k})_+ \\ &\geq \frac{1}{2z}\sum_k (z^2 - \mu_k)_+ \\ &= \frac{1}{2z} R_1^{\mathcal{F}}(z^2, \Delta_N), \end{aligned}$$

where $R_1^{\mathcal{F}}(z^2, \Delta_N)$ is the Riesz mean of the Neumann Laplace eigenvalues on $\mathcal{F}$. We can use Harrell–Stubbe's result [8] on lower bounds for $R_1^{\mathcal{F}}(z^2, \Delta_N)$ to get

$$\begin{aligned} R_1^{\mathcal{F}}(z^2, \Delta_N) = \sum_k (z^2 - \mu_k)_+ \;\; &\geq \;\; L_{1,n-1}^{cl}|\mathcal{F}|z^{n+1} + \frac{1}{4}\, L_{1,n-2}^{cl}\frac{|\mathcal{F}|}{\delta_{\mathbf{v}}(\mathcal{F})}\, z^n \\ &\quad -\frac{1}{96}\,(2\pi)^{2-n}\omega_n\frac{|\mathcal{F}|}{\delta_{\mathbf{v}}(\mathcal{F})^2}\, z^{n-1}, \end{aligned}$$

where $\delta_{\mathbf{v}}(\mathcal{F})$ is the width of $\mathcal{F}$ in the direction of $\mathbf{v} \in \mathbb{R}^{n-1}$ and

$$L_{1,n-1}^{cl} := \frac{1}{(4\pi)^{\frac{n-1}{2}}\Gamma(1+\frac{n+1}{2})}. \tag{2.9}$$

Comparing $L_{1,n-1}^{cl}$ with $C_{n,1}$ we have

$$L_{1,n-1}^{cl} = \frac{2n}{n+1}C_{n,1}. \tag{2.10}$$

Therefore

$$\begin{aligned} R_1(z) \;\; &\geq \;\; \frac{n}{n+1}C_{n,1}|\mathcal{F}|z^n + \frac{1}{8}\, L_{1,n-2}^{cl}\frac{|\mathcal{F}|}{\delta_{\mathbf{v}}(\mathcal{F})}\, z^{n-1} \\ &\quad -\frac{1}{192}\,(2\pi)^{2-n}\omega_n\frac{|\mathcal{F}|}{\delta_{\mathbf{v}}(\mathcal{F})^2}\, z^{n-2}. \end{aligned}$$

By Lemma 2.1, this bound holds for any proper subset Ω of $\mathcal{F} \times (-h, 0)$ with $\partial\Omega = \mathcal{F} \cup \mathcal{B}$. It gives a two–term lower bound only depending on the geometry of $\mathcal{F}$. When $n \to \infty$, the coefficient of the leading term tends to the optimal constant $C_{n,1}$.

We now prove the main theorem.

Proof of Theorem 2.3. The proof follows from Lemma 2.1 by choosing a suitable family of test functions. Consider the family of harmonic functions

$$f_{\xi'}(x) = e^{ix'\xi' + x_n|\xi'|}$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\xi' \in \mathbb{R}^{n-1}$. Replacing in (2.4) with $M = \mathbb{R}^{n-1}$ and $M_0 = \{|\xi'| \le z\}$ we get

$$\sum_j (z-\nu_j)_+ \int_{\mathbb{R}^{n-1}} |\hat{\varphi}_j(\xi')|^2 \, d\xi' \ge |\mathcal{F}| \int_{|\xi'|\le z} (z-|\xi'|)\, d\xi' - \int_{|\xi'|\le z} \int_{\mathcal{B}} \langle \mathbf{n}, (0,|\xi'|)\rangle e^{2x_n|\xi'|}\, ds d\xi',$$

where $\hat{\varphi}_j(\xi') = \int_{\mathcal{F}} e^{ix'\xi'} \varphi_j(x') ds$ is the Fourier transform of $\varphi_j|_{\mathcal{F}}$. Therefore,

$$R_1(z) = \sum_j (z-\nu_j)_+ \ge \frac{\omega_{n-1}}{n(2\pi)^{n-1}} |\mathcal{F}| z^n - \frac{(n-1)\omega_{n-1}}{(2\pi)^{n-1}} \int_0^z \int_{\mathcal{B}} \langle \mathbf{n}, \mathbf{e_n}\rangle e^{2x_n r} r^{n-1}\, ds dr,$$

where ω_{n-1} is the volume of a unit ball in $\mathbb{R}^{n-1}$. Proceeding with the Riesz iteration, we obtain inequality (2.5). This completes the proof. $\square$

We end this subsection with an example.

Example 2.6. Let consider the cone

$$\mathcal{C} := \{(x,y,z) : \tan^2(\alpha)(x^2+y^2) = (z+h)^2, \quad z \in (-h,0)\} \subset \mathbb{R}^3,$$

where α is the interior angle between $\mathcal{B}$ and the free surface $\mathcal{F} = \mathcal{C} \cap \mathbb{R}^2 \times \{0\}$. Computing (2.6) we obtain

$$\begin{aligned} A_{3,1}(z) &= \cos(\alpha) \int_0^z \int_0^{\frac{h}{|\tan(\alpha)|}} \frac{1}{|\cos(\alpha)|} e^{-2|\tan(\alpha)|tr} tr^2 \, dtdr \\ &= \frac{1}{4|\tan(\alpha)|\tan(\alpha)} \int_0^z \left(1 - e^{-2hr} - 2hre^{-2hr}\right) dr \\ &= \frac{1}{4|\tan(\alpha)|\tan(\alpha)} \left(z - \frac{1}{2h}(1 - e^{-2hz} + ze^{-2hz}) + \frac{1}{4h^2}(1-e^{-2hz})\right). \end{aligned}$$

Hence,

$$R^{\mathcal{C}}_{3,1}(z) \ge \frac{1}{12\pi} |\mathcal{F}| z^3 + \frac{1}{4|\tan(\alpha)|\tan(\alpha)} z + c,$$

where $c = A_{3,1}(z) - \frac{1}{4|\tan(\alpha)|\tan(\alpha)} z$.
One can ask if we can improve the power of z in the second term.

2.1. **Riesz means of sloshing problem on domains in $\mathbb{R}^2$.** In this section, we prove Theorem 1.1. Let us begin with the example which will be used in the proof of Theorem 1.1.

Example 2.7 (Triangular domain). Let $\Omega \subset \mathbb{R}^2$ be a triangle with interior angles $\alpha, \beta \in (0,\pi)$ as shown in Figure 2 (note that all the following calculations remain the same if one considers a trapezoid). Segment $\overline{OQ}$ is the free part $\mathcal{F}$ of the boundary with length L, and $\mathcal{B} = \overline{OP} \cup \overline{PQ}$. Replacing in (2.6) we have

$$A_{2,1}(z) = -\frac{1}{\pi} \int_0^z \int_{\overline{OP}\cup\overline{PQ}} \langle \mathbf{n}, (0,1)\rangle r e^{2yr}\, ds dr.$$

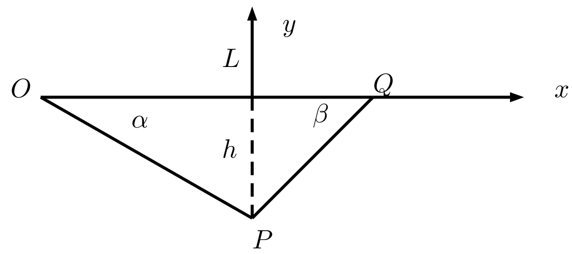

FIGURE 2.

First we calculate the above integral for $\alpha, \beta \in (0,\pi) \setminus \{\frac{\pi}{2}\}$

$$-\int_0^z \int_{\overline{OP}} \langle \mathbf{n}, (0,r)\rangle r e^{2yr}\, ds dr = \int_0^z r\cos(\alpha) \left(\int_0^{\frac{h}{|\tan(\alpha)|}} e^{-2x|\tan(\alpha)|r} \frac{1}{|\cos(\alpha)|}\, dx \right) dr$$
$$= \int_0^z \frac{1}{2\tan(\alpha)} \left(1 - e^{-2hr}\right) dr$$
$$= \frac{1}{2\tan(\alpha)} \left(z - \frac{1}{2h}(1 - e^{-2hz}) \right).$$

Similarly, we get

$$-\int_0^z \int_{\overline{PQ}} \langle \mathbf{n}, (0,r)\rangle e^{2yr}\, ds dr = \frac{1}{2\tan(\beta)} \left(z - \frac{1}{2h}(1 - e^{-2hz}) \right).$$

If α or β is equal to $\frac{\pi}{2}$ then $\langle \mathbf{n}, (0,1)\rangle = 0$ on $\overline{OP}$ or $\overline{PQ}$ respectively. We make a convention that $\frac{1}{\tan(\frac{\pi}{2})} = 0$. Therefore, we get an explicit formula for $A_{2,1}(z)$ in terms of interior angles $\alpha, \beta \in (0,\pi)$:

$$A_{2,1}(z) = \frac{1}{2\pi} \left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)} \right) \left(z - \frac{1}{2h}(1 - e^{-2hz}) \right).$$

We conclude

$$R_1(z) \geq \frac{1}{2\pi}|\mathcal{F}|z^2 + \frac{1}{2\pi} \left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)} \right) \left(z - \frac{1}{2h}(1 - e^{-2hz}) \right). \tag{2.11}$$

Proof of Theorem 1.1. By Lemma 2.2, we know

$$R^{\Omega}_{2,\gamma}(z) \geq R^{\tilde{\Omega}}_{2,\gamma}(z).$$

Having Theorem 2.3, it is enough to estimate $A^{\tilde{\Omega}}_{2,1}(z)$. Consider Figure 1. By assumption $\overline{PP_1}$ and $\overline{QQ_1}$ are line segments and $\tilde{\mathcal{B}}_c = \tilde{\mathcal{B}} \setminus (\overline{PP_1} \cup \overline{QQ_1})$.

$$A^{\tilde{\Omega}}_{2,1}(z) = -\frac{1}{\pi} \int_0^z \int_{\overline{PP_1} \cup \overline{QQ_1} \cup \tilde{\mathcal{B}}_c} \langle \mathbf{n}, (0,r)\rangle e^{2yr}\, ds dr.$$

Following the same calculation as in Example 2.7 we obtain

$$-\frac{1}{\pi} \int_0^z \int_{\overline{PP_1} \cup \overline{QQ_1}} \langle \mathbf{n}, (0,r)\rangle e^{2yr}\, ds dr = \frac{1}{2\pi} \left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)} \right) \left(z - \frac{1}{2\delta}(1 - e^{-2\delta z}) \right).$$

In order to compute the remaining term, let

$$\tilde{\mathcal{B}}_c^+ := \{(x,y) \in \tilde{\mathcal{B}} \,:\, \langle \mathbf{n}, (0,1)\rangle = \cos(\theta(x,y)) \geq 0\}$$

and

$$\tilde{\mathcal{B}}_c^- := \{(x,y) \in \tilde{\mathcal{B}} \,:\, \langle \mathbf{n}, (0,1)\rangle = \cos(\theta(x,y)) < 0\},$$

where $\theta(x,y)$ is the angle between $\mathbf{n}$ and the $(0,1)$. Hence

$$\begin{aligned}
-\frac{1}{\pi}\int_0^z \int_{\tilde{\mathcal{B}}_c^+ \cup \tilde{\mathcal{B}}_c^-} \langle \mathbf{n}, (0,1)\rangle r e^{2yr}\, ds dr \quad &\geq \quad -\frac{1}{\pi}\int_0^z \int_{\tilde{\mathcal{B}}_c^+} r\cos(\theta(x,y)) e^{-2\delta r}\, ds dr \\
&\quad\; +\frac{1}{\pi}\int_0^z \int_{\mathcal{B}_c^-} r|\cos(\theta(x,y))| e^{-2hr}\, ds dr \\
&= \quad -\frac{1}{2\pi\delta}\left(\frac{1}{2\delta} - \frac{1}{2\delta}e^{-2\delta z} - z e^{-2\delta z}\right)\int_{\tilde{\mathcal{B}}_c^+} \cos(\theta(x,y)) ds \\
&\quad\; \frac{1}{2\pi h}\left(\frac{1}{2h} - \frac{1}{2h}e^{-2h_z} - z e^{-2hz}\right)\int_{\mathcal{B}_c^-} |\cos(\theta(x,y))| ds \\
&\geq \quad -\frac{1}{2\pi\delta}\left(\frac{1}{2\delta} - \frac{1}{2\delta}e^{-2\delta z} - z e^{-2\delta z}\right)|\tilde{\mathcal{B}}_c|.
\end{aligned}$$

Hence

$$R_{2,1}^{\Omega}(z) \geq \frac{1}{2\pi}|\mathcal{F}|z^2 + \frac{1}{2\pi}\left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right) z + c,$$

where

$$c = \frac{1}{4\pi\delta}\left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right)(1 - e^{-2\delta z}) - \frac{1}{2\pi\delta}\left(\frac{1}{2\delta} - \frac{1}{2\delta}e^{-2\delta z} - z e^{-2\delta z}\right)|\tilde{\mathcal{B}}_c|.$$

Applying the Riesz iteration on both sides of the inequality completes the proof. □

It is clear from the proof of Theorem 1.1 that it is not necessary to assume that $\mathcal{F}$ is connected in the statement.
If in Theorem 1.1 $\alpha = \beta = \frac{\pi}{2}$, then $A_{2,\gamma}(z) = O(z^{\gamma-1})$. Hence, the power of z in the second term of lower bound (1.6) is not optimal. However, for a rectangular domain we can do a more explicit computation of its Reisz means and get a two-term asymptotically sharp lower bound. It immediately leads to the same bound on domains satisfying the John condition.

Proposition 2.8. *Let Ω be a bounded domain in $\mathbb{R}^2$ as in (1.1) with free part $\mathcal{F} = (0,\ell) \times \{0\}$. Assume that Ω satisfies the John condition then*

$$R_{\gamma}^{\Omega}(z) \geq \frac{\ell}{\pi(\gamma+1)} z^{\gamma+1} + \frac{1}{2} z^{\gamma}.$$

Proof. By Lemma 2.2, it is enough to prove the inequality for $\mathcal{R} = (0,\ell) \times (-h_\Omega, 0)$ where h_Ω is the depth of Ω. With the notation of Remark 2.5, we have $\mu_k = \frac{k^2\pi^2}{\ell^2}$, $k \in \mathbb{Z}_+$, and

$$\nu_k(\mathcal{R}) = \frac{k\pi}{\ell}\tanh\left(\frac{k\pi}{\ell} h_\Omega\right).$$

Hence, we have

$$R_1^{\mathcal{R}}(z) = \sum_k (z - \nu_k)_+ = \sum_k \left(z - \frac{k\pi}{\ell}\tanh\left(\frac{k\pi}{\ell}h_\Omega\right)\right)_+ \geq \sum_k \left(z - \frac{k\pi}{\ell}\right)_+.$$

We now use the following simple Lemma.

Lemma 2.9. *For any $x \geq 0$ and $k \in \mathbb{Z}_+$ we have*

$$\frac{1}{2}\left(x^2 + x\right) \leq \sum_{k\geq 0}(x-k)_+ \leq \frac{1}{2}\left(x^2 + x + 1\right).$$

Proof. The statement follows by a simple calculation.

$$\begin{aligned}\sum_{k\geq 0}(x-k)_+ &= x + x[x] - \frac{[x]^2}{2} - \frac{[x]}{2} \\ &= \frac{1}{2}\left(x^2+x\right) + \frac{1}{2}(x-[x])(1-x+[x])).\end{aligned}$$

We conclude by

$$0 \leq \frac{1}{2}(x-[x])(1-x+[x])) \leq \frac{1}{2}.$$

□

Using the above lemma, we get

$$R_1(z) = \frac{\pi}{\ell}\sum_{k\geq 0}(\frac{\ell}{\pi}z-k)_+ \geq \frac{\ell}{2\pi}z^2 + \frac{1}{2}z.$$

This completes the proof. □

2.2. **Bound on sum of eigenvalues.** We state and prove a more general version of Theorem 1.4.

Theorem 2.10. *For $n \geq 2$ the eigenvalues of problem* (1.2) *satisfy*

$$\frac{1}{k}\sum_{j=1}^{k}\nu_j \;\leq\; \frac{n-1}{n}\left(W_{n,k} - \frac{1}{W_{n,k}}(\nu_{k+1}-W_{n,k})^2\right) + W_{n,k}^{-(n-1)}c_{\mathcal{B}}(\nu_{k+1}),$$

where

$$c_{\mathcal{B}}(t) := (n-1)\omega_{n-1}|\mathcal{F}|^{-1}\int_0^t\int_{\mathcal{B}}\langle \mathbf{n},(0,1)\rangle r^{n-1}e^{2x_n r}ds\,dr$$

and $W_{n,k} = 2\pi\omega_{n-1}^{-\frac{1}{n-1}}\left(\frac{k}{|\mathcal{F}|}\right)^{\frac{1}{n-1}}$.

Proof. The proof follows the same lines of argument as in [8]. For the sake of completeness we present the whole argument.

By taking $z = \nu_{k+1}$, $M = \mathbb{R}^{n-1}$ and $M_0 = B_t$ in Lemma 2.1, where B_t is the ball of radius t centered at origin, we obtain

$$\nu_{k+1}t^{n-1} - \frac{n-1}{n}t^n \leq W_{n,k}^{n-1}\left(\nu_{k+1} - \frac{1}{k}\sum_{j=1}^{k}\nu_j\right) + c_{\mathcal{B}}(t). \tag{2.12}$$

Remark 2.11. One can immediately get a counterpart of Kröger's inequality for the eigenvalues of the sloshing problem by setting $t^{n-1} = c_n(k+1)$ in inequality (2.12):

$$\sum_{j=1}^{k+1}\nu_j \leq \frac{n-1}{n}c_n^{-\frac{1}{n-1}}(k+1)^{\frac{n}{n-1}} + (2\pi)^{1-n}\int_{B_t}\int_{\mathcal{B}}\langle \mathbf{n},(0,1)\rangle|\xi'|e^{2x_n|\xi'|}ds d\xi'.$$

To simplify (2.12), take $t = W_{n,k}x$. Then

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\nu_j - \frac{n-1}{n}W_{n,k} \;&\leq\; \frac{n-1}{n}W_{n,k}\left(x^n - \frac{n}{n-1}\frac{\nu_{k+1}}{W_{n,k}}x^{n-1} + \frac{n}{n-1}\frac{\nu_{k+1}}{W_{n,k}} - 1\right) \\ &\quad + W_{n,k}^{-(n-1)}c_{\mathcal{B}}(W_{n,k}x).\end{aligned}$$

If now we take $x = \frac{\nu_{k+1}}{W_{n,k}}$, then

$$\frac{1}{k}\sum_{j=1}^{k}\nu_j - \frac{n-1}{n}W_{n,k} \;\leq\; \frac{1}{n}W_{n,k}\left(nx-(n-1)-x^n\right) + W_{n,k}^{-(n-1)}c_{\mathcal{B}}(\nu_{k+1}).$$

Using the refinement of the Young inequalities stated in [8, Appendix A], for any $n \geq 2$ and every $x > 0$, the following inequality holds

$$nx-(n-1)-x^n \leq -(n-1)(x-1)^2.$$

Thus we conclude

$$\frac{1}{k}\sum_{j=1}^{k}\nu_j - \frac{n-1}{n}W_{n,k} \;\leq\; -\frac{n-1}{n}\frac{1}{W_{n,k}}(\nu_{k+1}-W_{n,k})^2 + W_{n,k}^{-(n-1)}c_{\mathcal{B}}(\nu_{k+1}).$$

□

Proof of Theorem 1.4. Under the assumption of Theorem 1.4 we have

$$\mathcal{B}^- := \{x \in \mathcal{B} \,:\, \langle \mathbf{n}, (0,r)\rangle \leq 0\} = \mathcal{B}.$$

Hence, $c_{\mathcal{B}}(x) \leq 0$ for any $x > 0$, and the statement follows from Theorem 2.10.

□

3. Mixed Steklov-Dirichlet eigenvalue problem

In this section we prove the results stated in Section 1.3. We begin by recalling the monotonicity property of the eigenvalues of Steklov–Dirichlet eigenvalue problem (1.5).

Lemma 3.1. [1, Proposition 3.1.1] *Let Ω and $\tilde{\Omega}$ be bounded domains of $\mathbb{R}^n$ whose boundaries $\partial\Omega = \mathcal{F}\cup\mathcal{B}$ and $\partial\tilde{\Omega} = \tilde{\mathcal{F}}\cup\tilde{\mathcal{B}}$ are as described in* (1.1). *Let Ω be a proper subset of $\tilde{\Omega}$ and $\tilde{\mathcal{F}} = \mathcal{F}$. Then the following inequality holds*

$$\eta_k(\Omega) > \eta_k(\tilde{\Omega}), \qquad \forall k \geq 1.$$

In particular,

$$R_\gamma^{\Omega}(z, \mathcal{D}_D) = \sum_j (z-\eta_j(\Omega))_+ \leq \sum_j (z-\eta_j(\tilde{\Omega}))_+ = R_\gamma^{\tilde{\Omega}}(z, \mathcal{D}_D).$$

Proof of Theorem 1.5. Since Ω is subset of the cylinder $\mathcal{F}\times[-h_\Omega, 0] =: \mathcal{R}$, where h_Ω is the depth of Ω, by Lemma 3.1 we have

$$R_\gamma^{\Omega}(z) \leq R_\gamma^{\mathcal{R}}(z).$$

For $\mathcal{R}$ the eigenvalues and eigenfunctions of problem (1.5) can be explicitly calculated (see [1]). They are of the form

$$\sqrt{\lambda_k}\coth(\sqrt{\lambda_k}h_\Omega),$$

where λ_k is the k-th Dirichlet eigenvalues of the Laplacian on $\mathcal{F}$. Hence, we have

$$\begin{aligned} R_1^{\mathcal{R}}(z) &= \sum_j (z-\eta_j)_+ = \sum_j (z-\sqrt{\lambda_j}\coth(\sqrt{\lambda_j}h_\Omega))_+ \\ &\leq \sum_j (z-\sqrt{\lambda_j})_+. \end{aligned}$$

The sequence $\sqrt{\lambda_j}$ of square root of eigenvalues of the Dirichlet Laplacian $-\Delta_D$ on $\mathcal{F}$ is equal to the eigenvalues of the Navier fractional Laplacian $(-\Delta)_N^{1/2}$ on $\mathcal{F}$. We denote its j-th eigenvalue by $\lambda_j((-\Delta)_N^{1/2})$. Musina and Nasarov in [18, 19, 20] studied the fractional Laplacian with Navier and Dirichlet type boundary conditions. Let us recall that for an arbitrary $s > 0$ the fractional Laplacian with Navier boundary conditions is defined by

$$(-\Delta^s_{Nv} f, f) = \sum_j \lambda_j^s |(f, \psi_j)|^2,$$

where ψ_j are eigenfunctions of the Dirichlet Laplacian $(-\Delta)_D$. The Dirichlet fractional Laplacian is defined by the closure from the class of functions $f \in C_0^\infty(\mathcal{F})$ of the quadratic form

$$((-\Delta)^s_D f, f) = \frac{1}{2\pi} \int |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi,$$

where $\hat{f}$ is the Fourier transform of f. It was proved in [18, Corollary 4] that for any $0 < s < 1$, the j-th eigenvalue $\lambda_j((-\Delta)^s_D)$ of the Dirichlet fractional Laplacian is strictly smaller than $\lambda_j((-\Delta)^s_{Nv})$ which implies

$$\lambda_j((-\Delta)^{1/2}_{Nv}) > \lambda_j((-\Delta)^{1/2}_D).$$

Therefore

$$\sum_j (z - \sqrt{\lambda_j})_+ = \sum_j \left(z - \lambda_j((-\Delta)^{1/2}_{Nv})\right)_+ < \sum_j \left(z - \lambda_j((-\Delta)^{1/2}_D)\right)_+ .$$

We now use the bound on the Riesz means of the Dirichlet fractional Laplacian $(-\Delta)^{1/2}_D$ proved by Laptev in [13, Corollary 2.3].

$$\sum_j \left(z - \lambda_j((-\Delta)^{1/2}_D)\right)_+ \le (2\pi)^{-(n-1)} |\mathcal{F}| z^n \left(\int_{\mathbb{R}^{n-1}} (1 - |\xi|)_+ d\xi\right) = C_{n,1} |\mathcal{F}| z^n.$$

This completes the proof. □

Remark 3.2. Applying the Laplace transform on inequality (1.14), we can get an immediate upper bound for the trace of the heat kernel of operator $\mathcal{D}_D$. More precisely

$$\sum_{j=0}^\infty e^{-\eta_j t} \le \frac{\Gamma(n)}{(4\pi)^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} \frac{|\mathcal{F}|}{t^{n-1}}.$$

Proof of Theorem 1.7. We first show that inequality (1.17) holds for a rectangular domain $\mathcal{R} = (0, \ell) \times (-h, 0)$. With the same notations as in the proof of Theorem 1.5, we have $\lambda_j((0,\ell)) = \frac{j^2\pi^2}{\ell^2}$, $j \in \mathbb{Z}_+$ and

$$\eta_j(\mathcal{R}) = \frac{j\pi}{\ell} \coth\left(\frac{j\pi}{\ell} h\right).$$

Thus

$$R_1^{\mathcal{R}}(z) = \sum_j (z - \eta_j)_+ = \sum_{j>0} \left(z - \frac{j\pi}{\ell} \coth\left(\frac{j\pi}{\ell} h\right)\right)_+ \le \sum_j \left(z - \frac{j\pi}{\ell}\right)_+ .$$

By Lemma 2.9 we have

$$\frac{\pi}{\ell} \sum_{j>0} \left(\frac{\ell}{\pi} z - j\right)_+ \le \frac{\pi}{2\ell} \left(\frac{\ell^2}{\pi^2} z^2 - \frac{\ell}{\pi} z + 1\right) = \frac{\ell}{2\pi} z^2 - \frac{1}{2} z + \frac{\pi}{2\ell}.$$

By monotonicity property in Lemma 3.1 the statement of the theorem follows. □

Remark 3.3. Let $\Omega \subset \mathbb{R}^2_-$ be a domain with $\partial\Omega = \mathcal{F} \cup \mathcal{B}$, $\mathcal{F} = (0, \ell)$, containing a rectangular domain $\mathcal{R} = (0, \ell) \times (-h, 0)$. We can get a two–term asymptotically sharp lower bound on $R_1^\Omega(z)$ with an optimal leading term. Indeed, using the monotonicity result together with Lemma 2.9 and the following inequality

$$x \coth(x) \leq 1 + x$$

for $z \geq 1$ we obtain

$$\begin{aligned} R_1^\Omega(z) \geq R_1^{\mathcal{R}}(z) &= \sum_{j>0} \left(z - \frac{j\pi}{\ell} \coth\left(\frac{j\pi}{\ell} h \right) \right)_+ \\ &\geq \sum_{j>0} \left(z - 1 - \frac{j\pi}{\ell} \right)_+ \\ &\geq \frac{\ell}{2\pi}(z-1)^2 - \frac{1}{2}(z-1) \\ &= \frac{\ell}{2\pi} z^2 - \left(\frac{1}{2} + \frac{\ell}{\pi} \right) z + \frac{1}{2}. \end{aligned}$$

4. Appendix: Two-term asymptotics for $R_\gamma(z)$

by Francesco Ferrulli and Jean Lagacé

In this appendix we obtain in dimension 2, under conditions slightly different than those of Theorem 1.1 two-term asymptotics (1.9) and (1.16) rather than lower and upper bounds for respectively the sloshing and the Steklov-Dirichlet problem. The conditions are those required for Theorems 1.2.2 and 1.3.2, and Propositions 1.2.6 and 1.3.5 of [16].

Let us start by introducing *local John's condition*.

Definition 4.1. A corner point V between $\mathcal{F}$ and $\mathcal{B}$ is said to satisfy *local John's condition* if there exists a neighbourhood $\mathcal{O}_V$ of V such that $\mathcal{O}_V \cap \mathcal{B} \subset \mathcal{F} \times (-\infty, 0)$.

We now prove the following theorem.

Theorem 4.2. *Let Ω be a simply connected bounded Lipschitz planar domain with the sloshing surface $\mathcal{F}$ of length L and walls $\mathcal{B}$ which are C^1-regular near the corner points A and B. Let α and β be the interior angles between $\mathcal{B}$ and $\mathcal{F}$ at the points A and B resp., and assume either that*

- $0 < \beta \leq \alpha < \pi/2$*; or*
- $0 < \beta < \alpha = \pi/2$ *and A satisfies local John's condition; or*
- $\beta = \alpha = \pi/2$ *and both A and B satisfy local John's condition.*

Then, the following two-term asymptotics for the Riesz mean of order $\gamma > 0$ hold as $z \to \infty$:

$$R_\gamma^\Omega(z, \mathcal{D}_N) = C_{2,\gamma} L z^{\gamma+1} + \frac{\pi}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) z^\gamma + o\left(z^\gamma\right), \tag{4.1}$$

and

$$R_\gamma^\Omega(z, \mathcal{D}_D) = C_{2,\gamma} z^{\gamma+1} - \frac{\pi}{8}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) z^\gamma + o\left(z^\gamma\right), \tag{4.2}$$

where $C_{2,\gamma} = ((1+\gamma)\pi)^{-1}$.

Proof. In order to be able to prove this statement in one proof for both problems at the same time, we will make the following notational convention : we write $\nu_k^+ := \eta_k$ for the eigenvalues of the Steklov-Dirichlet problem and $\nu_k^- := \nu_k$ for those of the sloshing problem. Similarly, we write $\mathcal{D}_+ := \mathcal{D}_D$ and $\mathcal{D}_- := \mathcal{D}_N$. The conditions we are assuming are exactly those of [16] that yield the following asymptotics for $\nu_k^\pm$:

$$\begin{aligned}\nu_k^\pm &= \frac{\pi}{L}\left(k-\frac{1}{2}\right) \pm \frac{\pi^2}{8L}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + r(k) \\ &= \frac{\pi}{L}k + C_\pm + r(k).\end{aligned} \tag{4.3}$$

Moreover, $r(k) = o(1)$. This allows us to write

$$R_\gamma^\Omega(z;\mathcal{D}_\pm) = \sum_{0\le\nu_k^\pm\le z}\left(z-\frac{\pi}{L}k - C_\pm - r(k)\right)^\gamma. \tag{4.4}$$

Observe that $\nu_k^\pm \le z$ if and only if

$$k + r(k) \le \frac{L}{\pi}\left(z - C_\pm\right). \tag{4.5}$$

Since $r(k) = o(1)$, there exists a function $s(z) = o(1)$ such that $\nu_k^\pm \le z$ if and only if $k \le g(z)$, for $g(z) := \frac{L}{\pi}(z - C_\pm) + s(z)$. Since we start counting eigenvalues at $k = 1$, this allows us to rewrite equation (4.4) as

$$R_\gamma^\Omega(z;\mathcal{D}_\pm) = \sum_{1\le k\le g(z)}\left(\frac{\pi}{L}\right)^\gamma \left(g(z) - k - s(z) - \tilde{r}(k)\right)^\gamma, \tag{4.6}$$

where $\tilde{r} = L\pi^{-1}r$. From now on we make use of the strategy of the proof by Lagacé and Parnovski of [12, Theorem 1.6]. Let us write $g(z) = a_z + \tau_z$ where a_z is the integer part and τ_z the fractional part, and rewrite the previous sum as

$$R_\gamma^\Omega(z;\mathcal{D}_\pm) = \left(\frac{\pi}{L}\right)^\gamma \sum_{0\le k\le a_z-1}(k+\tau_z - s(z) - \tilde{r}(k))^\gamma. \tag{4.7}$$

Since $\tilde{r}(k) = o(1)$, there exists some K such that for all $k > K$, $\tilde{r}(k) < 1/4$. We split the sum into

$$\sum_{0\le k\le a_z-1}(k+\tau_z - s(z) - \tilde{r}(k))^\gamma = \left(\sum_{0\le k\le K} + \sum_{K<k\le a_z-1}\right)(k+\tau_z - s(z) - \tilde{r}(k))^\gamma. \tag{4.8}$$

We have that

$$\begin{aligned}\sum_{0\le k\le K}(k+\tau_z - s(z) - \tilde{r}(k))^\gamma &\le K\left(K+1+o(z)+\inf_{k\le K} r(k)\right)^\gamma \\ &= o(z^\gamma).\end{aligned} \tag{4.9}$$

For the second sum, suppose that we have chosen z large enough that $s(z) < K/4$. Then, we have that

$$\begin{aligned}\sum_{K<k\le a_z-1} (k+\tau_z - s(z) - \tilde r(k))^\gamma &= \sum_{K<k\le a_z-1} (k+\tau_z)^\gamma \left(1 - \frac{s(z)-\tilde r(k)}{k+\tau_z}\right)^\gamma \\ &= \sum_{K<k\le a_z-1} \left((k+\tau_z)^\gamma - s(z)\, O\left(k^{\gamma-1}\right) + o\left(k^{\gamma-1}\right)\right), \\ &= o\left(z^\gamma\right) + \sum_{K<k\le a_z-1} (k+\tau_z)^\gamma\end{aligned} \tag{4.10}$$

Finally, the Euler-Maclaurin formula tells us that

$$\begin{aligned}\sum_{K<k\le a_z-1} (k+\tau_z)^\gamma &= \int_{K+1}^{a_z-1} (k+\tau_z)^\gamma \,\mathrm{d}k + \frac{1}{2}\left((K+1+\tau_z)^\gamma + (g(z)-1)^\gamma\right) + O\left(z^{\gamma-1}\right) \\ &= \int_{K+1+\tau_z}^{g(z)-1} k^\gamma \,\mathrm{d}k + \frac{1}{2}\left((K+1+\tau_z)^\gamma + (g(z)-1)^\gamma\right) + O\left(z^{\gamma-1}\right) \\ &= \frac{1}{\gamma+1}(g(z)-1)^{\gamma+1} + (g(z)-1)^\gamma + O(1) + O\left(z^{\gamma-1}\right)\end{aligned} \tag{4.11}$$

We reexpand $g(z)$ and $C_\pm$ and collect all terms together to obtain directly that

$$R_\gamma^\Omega(z;\mathcal{D}_N) = C_{2,\gamma} L z^{\gamma+1} + \frac{\pi}{8}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) z^\gamma + o\left(z^\gamma\right), \tag{4.12}$$

and

$$R_\gamma^\Omega(z;\mathcal{D}_D) = C_{2,\gamma} L z^{\gamma+1} - \frac{\pi}{8}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) z^\gamma + o\left(z^\gamma\right) \tag{4.13}$$

finishing the proof. □

References

[1] R. Bañuelos, T. Kulczycki, I. Polterovich, and B. Siudeja. Eigenvalue inequalities for mixed Steklov problems. In *Operator theory and its applications*, volume 231 of *Amer. Math. Soc. Transl. Ser. 2*, pages 19–34. Amer. Math. Soc., Providence, RI, 2010.

[2] C. Bandle. *Isoperimetric inequalities and applications*, volume 7 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.

[3] K. Chandrasekharan and S. Minakshisundaram. *Typical means*. Oxford University Press, 1952.

[4] A. El Soufi, E. M. Harrell, S. Ilias, and J. Stubbe. On sums of eigenvalues of elliptic operators on manifolds. arXiv:1507.02632.

[5] D. W. Fox and J. R. Kuttler. Sloshing frequencies. *Z. Angew. Math. Phys.*, 34(5):668–696, 1983.

[6] L. Geisinger, A. Laptev, and T. Weidl. Geometrical versions of improved Berezin-Li-Yau inequalities. *J. Spectr. Theory*, 1(1):87–109, 2011.

[7] E. M. Harrell and L. Hermi. On Riesz means of eigenvalues. *Comm. Partial Differential Equations*, 36(9):1521–1543, 2011.

[8] E. M. Harrell and J. Stubbe. Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian. arXiv:1607.02207.

[9] V. Ivrii. Spectral asymptotics for Dirichlet to Neumann operator, 2018. arXiv:1802.07524.

[10] P. Kröger. Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. *J. Funct. Anal.*, 106(2):353–357, 1992.

[11] N. Kuznetsov, T. Kulczycki, M. Kwaśnicki, A. Nazarov, S. Poborchi, I. Polterovich, and B. Siudeja. The legacy of Vladimir Andreevich Steklov. *Notices Amer. Math. Soc.*, 61(1):9–22, 2014.

[12] J. Lagacé and L. Parnovski. A generalised Gauss circle problem and integrated density of states *J. Spectr. Theory*, 6(4):859–879, 2016.

[13] A. Laptev. Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Funct. Anal.*, 151(2):531–545, 1997.

[14] Ari Laptev and Timo Weidl. Recent results on Lieb-Thirring inequalities. In *Journées "Équations aux Dérivées Partielles" (La Chapelle sur Erdre, 2000)*, pages Exp. No. XX, 14. Univ. Nantes, Nantes, 2000.

[15] Ari Laptev and Timo Weidl. Sharp Lieb-Thirring inequalities in high dimensions. *Acta Math.*, 184(1):87–111, 2000.

[16] M. Levitin, L. Parnovski, I. Polterovich, and D. A. Sher. Sloshing, Steklov and corners I: Asymptotics of sloshing eigenvalues, 2017. arXiv:1709.01891.

[17] P. Li and S. T. Yau. On the Schrödinger equation and the eigenvalue problem. *Comm. Math. Phys.*, 88(3):309–318, 1983.

[18] R. Musina and A. I. Nazarov. On fractional Laplacians. *Commun. Partial Differ. Equations*, 39(9):1780–1790, 2014.

[19] R. Musina and A. I. Nazarov. On fractional Laplacians. II. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 33(6):1667–1673, 2016.

[20] R. Musina and A. I. Nazarov. On fractional Laplacians. III. *ESAIM, Control Optim. Calc. Var.*, 22(3):832–841, 2016.

[21] L. Provenzano and J. Stubbe. Weyl–type bounds for steklov eigenvalues. J. Spectr. Theory, to appear.

[22] L. Sandgren. A vibration problem. *Medd. Lunds Univ. Mat. Sem.*, 13:1–84, 1955.

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