ON A SHARP HARDY INEQUALITY FOR CONVEX DOMAINS

FARIT AVKHADIEV AND ARI LAPTEV

Abstract. The aim of this short note is to obtain a sharp Hardy inequality for convex domains involving both the distance to the boundary and the distance to the origin. In particular this would imply a Hardy-Sobolev inequality for the class of symmetric functions in a ball.

Key-words: Hardy inequalities; Sobolev inequalities *MSC (2000):* Primary: 35P15; Secondary: 81Q10

0. Introduction

The classical Hardy inequality states that if $d \geq 3$ then for any function u such that $\nabla u \in L^2(\mathbb{R}^d)$

$$\left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx. \tag{1}$$

The constant $((d-2)/2)^2$ in (1) is sharp but it is not achieved.

Hardy's inequality for convex domains in $\mathbb{R}^d$ is usually given in terms of the distance to the boundary. Namely, let $\Omega \subset \mathbb{R}^d$ be a convex domain and let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ be the distance from $x \in \Omega$ to the boundary $\partial\Omega$. The following Hardy inequality is well known (see [5], [6])

$$\int_\Omega |\nabla u|^2\, dx \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx, \qquad u \in H_0^1(\Omega). \tag{2}$$

In this case the equality is also not achieved. This gives room for various improvements of the later inequality. It has been shown by H. Bresis and M. Markus in [4] that

$$\int_\Omega |\nabla u|^2\, dx \geq \frac{1}{4} \int_\Omega \frac{|u(x)|^2}{\delta^2(x)}\, dx + \lambda(\Omega) \int_\Omega |u(x)|^2\, dx, \qquad u \in H_0^1(\Omega),$$

where

$$\lambda(\Omega) \geq \frac{1}{4\mathrm{diam}^2(\Omega)}.$$

Date: September 3, 2009.

In [11] M. Hoffmann-Ostenhof, Th. Hoffmann-Ostenhof and A. Laptev estimated this constant by the Lebesgue measure of Ω

$$\lambda(\Omega) \geq \frac{3}{4}\Big(\frac{v_d}{|\Omega|}\Big)^{2/d},$$

where v_d is the volume of unit ball.
F. Avkhadiev [1], [2] and S. Filippas, V. Maz'ya and A. Tertikas [8] have proved that

$$\lambda(\Omega) \geq 3D_{\rm int}^{-2}(\Omega),$$

where $D_{\rm int}(\Omega) = 2\sup\{\delta(x) : x \in \Omega\}$ is the interior diameter of Ω.
Recently F. Avkhadiev and K.-J. Wirths [3] have proved that $\lambda(\Omega) \geq 4\lambda_0 D_{\rm int}^{-2}(\Omega)$, where $\lambda_0 = 0.940...$ is defined as the first positive root of the equation $J_0(\lambda)+2\lambda J_0'(\lambda) = 0$ for Bessel's function and this constant is sharp for all dimensions.

The main purpose of this short article is to prove the following result which is a certain generalisation of Theorem 3.2 [9]:

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain and let $B_\rho = \{x : |x| \leq \rho\} \subset \Omega$. For $x \in \Omega \setminus B_\rho$ denote by $\delta_\rho(x) = {\rm dist}\,(x, \partial B_\rho) = |x| - \rho$ and $\delta_\Omega = {\rm dist}\,(x, \partial\Omega)$. Then for any $u \in H_0^1(\Omega \setminus B_\rho)$*

$$\int_{\Omega\setminus B_\rho} |\nabla u(x)|^2\, dx \geq \frac{1}{4}\int_{\Omega\setminus B_\rho}\Big(\frac{(d-1)(d-3)}{|x|^2} + \frac{1}{\delta_\rho^2(x)} + \frac{1}{\delta_\Omega^2(x)} - 2\,\frac{\Delta\delta_\Omega(x)}{\delta_\Omega(x)} - 2\,\frac{x\cdot\nabla\delta_\Omega(x)}{|x|\delta_\rho(x)\delta_\Omega(x)} + 2(d-1)\,\frac{x\cdot\nabla\delta_\Omega(x)}{|x|^2\delta_\Omega(x)}\Big)|u(x)|^2\, dx.$$

This result implies several corollaries which have some independent interest.

Corollary 1. *Let $\Omega = B_R = \{x : |x| < R\}$, $R > \rho$, and let $\delta_R = \delta_\Omega = {\rm dist}\,(x, \partial B_R) = R - |x|$. Then for any $u \in H_0^1(\Omega \setminus B_\rho)$*

$$\int_{B_R\setminus B_\rho} |\nabla u(x)|^2\, dx \geq \frac{1}{4}\int_{B_R\setminus B_\rho}\Big(\frac{(d-1)(d-3)}{|x|^2} + \frac{1}{\delta_\rho^2(x)} + \frac{1}{\delta_R^2(x)} + \frac{2}{\delta_\rho(x)\delta_R(x)}\Big)|u(x)|^2\, dx.$$

Proof. This statement follows from Theorem 1, since

$$-\frac{\Delta\delta_R(x)}{\delta_R(x)} = \frac{d-1}{|x|\delta_R(x)} \qquad \text{and} \qquad x\cdot\nabla\delta_R(x) = -|x|.$$

□

Remark 1. *In particular, if $R \to \infty$, then Corollary 1 provides us with the Hardy inequality for exterior of the ball B_ρ*

$$\int_{\mathbb{R}^d \setminus B_\rho} |\nabla u(x)|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^d \setminus B_\rho} \Big(\frac{(d-1)(d-3)}{|x|^2} + \frac{1}{\delta_\rho^2(x)} \Big) |u(x)|^2 \, dx.$$

Remark 2. *It has been noticed in* [7] *that the inequality* (2) *holds for a not necessary convex conical sector $\Omega = \Omega_\alpha \subset \mathbb{R}^2$ of angle $\alpha \leq \pi + 4\arctan(4\Gamma^2(3/4)\Gamma^{-2}(1/4))$, whereas Corollary 1 shows that* (2) *is also true for $\Omega = B_R \setminus B_\rho$. It would be interesting to describe a class of non-convex domains for which* (2) *remains true.*

If we let $\rho \to 0$, then using $(d-1)(d-3) + 1 = (d-2)^2$ we find

Corollary 2.

$$\int_{B_R} |\nabla u(x)|^2 \, dx \geq \frac{1}{4} \int_{B_R} \Big(\frac{(d-2)^2}{|x|^2} + \frac{1}{\delta_R^2(x)} + \frac{2}{|x|\delta_R(x)} \Big) |u(x)|^2 \, dx,$$

where $u \in H_0^1(B_R)$ if $d \geq 2$ and $u \in H_0^1(B_R \setminus \{0\})$ if $d = 1$.

Remark 3. *Note that the latter inequality without the extra positive term $2/(|x|\delta_R(x))$ follows from Theorem 3.2* [9]*. Note that both constants at the first two terms in the right hand side of this inequality are sharp.*

If we now assume that $-\Delta\delta_\Omega(x) + (d-2)\, x \cdot \nabla\delta_\Omega(x)/|x|^2 \geq 0$ and $\rho = 0$ we recover Theorem 3.2 [9].

Corollary 3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain and let $B_\rho = \{x : |x| \leq \rho\} \subset \Omega$ and let us assume that $-\Delta\delta_\Omega(x) + (d-2)\, x \cdot \nabla\delta_\Omega(x)/|x|^2 \geq 0$. Then for any $u \in H_0^1(\Omega \setminus \{0\})$ we have*

$$\int_{\Omega \setminus B_\rho} |\nabla u(x)|^2 \, dx \geq \frac{1}{4} \int_{\Omega \setminus B_\rho} \Big(\frac{(d-2)^2}{|x|^2} + \frac{1}{\delta_\Omega^2(x)} \Big) |u(x)|^2 \, dx.$$

Corollary 4. *Let Ω be a bounded convex domain and let us assume that the origin is chosen such that $\max\{\delta_\Omega(x) : x \in \Omega\}$ is achieved at 0. Suppose also that $B_\rho \subset \Omega \subset \{x : (d-2)|x| < (d-1)\rho\}$, $d \geq 2$. Then for any $u \in H_0^1(\Omega \setminus B_\rho)$*

$$\int_{\Omega \setminus B_\rho} |\nabla u(x)|^2 \, dx \geq \frac{1}{4} \int_{\Omega \setminus B_\rho} \Big(\frac{(d-1)(d-3)}{|x|^2} + \frac{1}{\delta_\rho^2(x)} + \frac{1}{\delta_\Omega^2(x)} \Big) |u(x)|^2 \, dx.$$

Proof. Lemma 3 implies that for convex domains $\Delta\delta_\Omega(x) \leq 0$. By using Lemma 2 we find that $x \cdot \nabla\delta_\Omega \leq 0$. Besides if $(d-2)|x| < (d-1)\rho$ then

$$\frac{2}{\delta_\rho(x)\delta_\Omega(x)} \geq \frac{2(d-1)}{|x|\delta_\Omega(x)}$$

and we obtain the statement. □

Remark 4. *In particular, if $d = 2$, then for any convex domain satisfying properties of Corollary 4 we have*

$$\int_{\Omega\setminus B_\rho} |\nabla u(x)|^2\, dx \geq \frac{1}{4}\int_{\Omega\setminus B_\rho}\Big(\frac{1}{(|x|-\rho)^2} - \frac{1}{|x|^2} + \frac{1}{\delta_\Omega^2(x)}\Big)|u(x)|^2\, dx$$

and if $\rho \to 0$, then we recover (2).

It has been noticed in the recent paper of R. Frank and R. Seiringer [10] (see Lemma 4.3) that for any non-negative symmetric decreasing function u in $\mathbb{R}^d$

$$\|u\|^2_{2^*,2} = v_d^{-2/d}\int_{\mathbb{R}^d}\frac{u^2(x)}{|x|^2}\, dx, \tag{3}$$

where $\|u\|_{2^*,2}$ is the Lorentz norm and $2^* = 2d/(d-2)$. Combining (3) with Corollary 2 we obtain

Theorem 2. *Let $B_R = \{x \in \mathbb{R}^d : |x| < R\}$, $d \geq 3$ and $\delta_R(x) = R - |x|$ be the distance from x to the boundary ∂B_R. Then for any non-negative symmetric decreasing function $u \in H_0^1(B_R)$ we have*

$$\int_{B_R}|\nabla u(x)|^2\, dx \geq \frac{1}{4}\int_{B_R}\frac{u^2(x)}{\delta_R^2(x)}\, dx + v_d^{2/d}\Big(\frac{(d-2)}{2}\Big)^2\|u\|^2_{2^*,2}. \tag{4}$$

Using that

$$\|u\|^2_{2^*} \leq \frac{d-2}{d}\|u\|^2_{2^*,2},$$

(see [10]) and (4) we obtain the following Hardy-Sobolev inequality with constants which are independent of the radius of the ball.

Corollary 5. *Under the assumptions of Theorem 2*

$$\int_{B_R}|\nabla u(x)|^2\, dx \geq \frac{1}{4}\Big[\int_{B_R}\frac{u^2(x)}{\delta_R^2(x)}\, dx + d(d-2)\, v_d^{2/d}\,\|u\|^2_{2^*}\Big].$$

1. Some auxiliary results

We begin with a statement which is a simple corollary of the Cauchy-Schwarz inequality and partial integration. Different versions of this statement could be found for example in [11] and [12].

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ and let $\mathcal{F}(x)$ be a vector field $x \in \Omega$. Then assuming that $\mathcal{F}(x)$ and $\operatorname{div}\mathcal{F}(x)$ are finite for $x \in \operatorname{supp} u$ we obtain*

$$\frac{1}{4}\int_\Omega\Big(2\operatorname{div}\mathcal{F}(x) - |\mathcal{F}(x)|^2\Big)|u(x)|^2\, dx \leq \int_\Omega |\nabla u|^2\, dx, \quad u \in C_0^\infty(\Omega). \tag{5}$$

Proof. Indeed

$$\Big(\int_\Omega \operatorname{div}\mathcal{F}(x)|u(x)|^2\,dx\Big)^2 = \Big(\int_\Omega \mathcal{F}(x)\cdot\nabla(|u(x)|^2)\,dx\Big)^2 \le 4\int_\Omega |\nabla u|^2\,dx \int_\Omega |\mathcal{F}(x)|^2|u(x)|^2\,dx.$$

Then it only remains to notice that

$$\frac14\int_\Omega \Big(2\operatorname{div}\mathcal{F}(x) - |\mathcal{F}(x)|^2\Big)|u(x)|^2\,dx \le \frac14 \frac{\Big(\int_\Omega \operatorname{div}\mathcal{F}(x)|u(x)|^2\,dx\Big)^2}{\int_\Omega |\mathcal{F}(x)|^2|u(x)|^2\,dx} \le \int_\Omega |\nabla u|^2\,dx.$$

□

We now prove a simple geometrical lemma which is valid for bounded convex domains.

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$ be a convex bounded domain with C^1 boundary. Then the value of the interior radius $R_{\rm int} = sup\,\{\delta_\Omega(x) : x\in\Omega\}$ is achieved at some point $x_0\in\Omega$ and for any $x\in\Omega$*

$$\nabla\delta_\Omega(x)\cdot(x-x_0)\le 0. \tag{6}$$

Proof. The function $\delta_\Omega : \overline{\Omega}\to\mathbb{R}_+$ defined on the compact set $\overline{\Omega}$ is continuous. Thus $R_{\rm int}$ is achieved at some $x_0\in\Omega$.
Let us now consider sets Ω_r, $0<r<R_{\rm int}$ such that

$$\Omega_r = \{x\in\Omega :\ \delta_\Omega(x) > r\}.$$

Domains Ω_r are convex and obviously $x_0\in\Omega_r$ for $0<r\le R_{\rm int}$. Then for any $x\in\partial\Omega_r$ the vector $-\nabla\delta_\Omega(x)$ is the unit outer normal to $\partial\Omega_r$ at x. Then obviously $-\nabla\delta_\Omega(x)\cdot(x-x_0)\ge 0$ which completes the proof. □

Remark 5. *The statement of Lemma 2 remains true for an arbitrary bounded convex domain. In this case at points where $\nabla\delta_\Omega(x)$ is not uniquely defined, one should consider in* (6) *a normal to any of the supporting hyperplanes to Ω_r at $x\in\partial\Omega_r$.*

The next result concerning the Laplacian of the distance function for convex domain is well known.

Lemma 3. *Let $\Omega\subset\mathbb{R}^d$, $d\ge 2$, be a convex domain and let δ_Ω be the distance function to its boundary. Then $-\Delta\,\delta_\Omega(x)$ is a positive measure.*

Proof. Let $y_0\in\Omega$ and assume that $\nabla\delta_\Omega(y_0)$ exists. Consider an orthornormal system of coordinates $\{y_1, y_2,\dots,y_{d-1}\}$ in the hyperplane defined by $y_0\in\Omega$ and by $-\nabla\delta_\Omega(y)$. In these coordinates the boundary $\partial\Omega$ could be represented by a concave function from the class Lip_1. Note that $\partial_{y_d}\delta_\Omega(y_0)=0$, where y_d is the coordinate along $\nabla\delta_\Omega(y)$. Using the fact that the Laplacian Δ is invariant under rotations and parallel transport we therefore obtain that $\Delta\delta_\Omega(y_0) \le 0$. □

2. Proof of the main result

Let

$$\mathcal{F}(x) = \frac{(d-1)x}{|x|^2} - \frac{\nabla\delta_\rho(x)}{\delta_\rho(x)} - \frac{\nabla\delta_\Omega(x)}{\delta_\Omega(x)}.$$

Then using that $|\nabla\delta_\rho(x)| = |\nabla\delta_\Omega(x)| = 1$ we obtain

$$\operatorname{div}\mathcal{F}(x) = \frac{(d-1)(d-2)}{|x|^2} + \frac{1}{\delta_\rho^2(x)} + \frac{1}{\delta_\Omega^2(x)} - \frac{(d-1)}{|x|\delta_\rho(x)} - \frac{\Delta\delta_\Omega(x)}{\delta_\Omega(x)}. \tag{7}$$

and since $\nabla\delta_\rho(x) = x/|x|$

$$|\mathcal{F}(x)|^2 = \frac{(d-1)^2}{|x|^2} + \frac{1}{\delta_\rho^2(x)} + \frac{1}{\delta_\Omega^2(x)} - 2\,\frac{(d-1)}{|x|\delta_\rho(x)} - 2\,\frac{(d-1)x\cdot\nabla\delta_\Omega(x)}{|x|^2\delta_\Omega(x)} + 2\,\frac{x\cdot\nabla\delta_\Omega(x)}{|x|\delta_\rho(x)\delta_\Omega(x)}. \tag{8}$$

Substituting (7) and (8) into the left hand side of (5) we complete the proof of Theorem 1.

Acknowledgments. The authors acknowledge a partial support by the SPECT ESF European programme. F.G. Avkhadiev was supported by the Russian Foundation for Basic Research (project no. 08-01-00381) and by the Göran Gustafssons Stiftelse in Sweden.

References

[1] F. G. Avkhadiev, *Hardy Type Inequalities in Higher Dimensions with Explicit Estimate of Constants*, Lobachevskii J. Math. **21** (2006), 3–31 (electronic, http://ljm.ksu.ru).

[2] F. G. Avkhadiev, *Hardy-type inequalities on planar and spatial open sets*, Proceeding of the Steklov Institute of Mathematics **255** (2006), no.1, 2–12 (translated from Trudy Matem. Inst. V.A. Steklova, 2006, v.255, 8–18).

[3] F. G. Avkhadiev, K.-J. Wirths, *Unified Poincaré and Hardy inequalities with sharp constants for convex domains*, Z. Angew. Math. Mech. **87** (2007), 632–642.

[4] H.Brezis and M. Marcus, Hardy's inequalities revisited. Dedicated to Ennio De Giorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 1-2 (1998), 217–237.

[5] E.B.Davies, Spectral theory and differential operators. Cambridge Studies in Advanced Mathematics, 42. Cambridge University Press, Cambridge, 1995. x+182 pp.

[6] E. B. Davies, *A Review of Hardy inequalities*, The Maz'ya anniversary Collection. Vol. 2. Oper. Theory Adv. Appl. **110** (1999), 55–67.

[7] E. B. Davies, *The Hardy constant*, Quart.J.math. Oxford (2) **46** (1995), 417–431.

[8] S. Fillippas, V. Maz'ya, A. Tertikas, *Sharp Hardy-Sobolev Inequalities* , C. R. Acad. Sci. Paris **339** (2004),no. 7, 483–486.

[9] S. Fillippas, L. Moschini, A. Tertikas, *Sharp twosided heat kernel estimates for critical Schrödinger operators on bounded domains* , Commun. Math. Phys. **273** (2007), 237281.

[10] R.L. Frank and R.Seiringer, *Non-linear ground state representations and sharp Hardy inequalities.* arXiv:0803.0503v1 [math.AP] 4 Mar 2008.

[11] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and A. Laptev, *A geometrical version of Hardy's inequalities*, J. Funct. Anal. **189** (2002), no 2, 539–548.

[12] M.Hoffmann-Ostenhof, Th.Hoffmann-Ostenhof, A.Laptev and J.Tidblom, *Many Particle Hardy Inequalities*, accepted by JLMS.

F. G. Avkhadiev: Chebotarev Research Institute, Kazan State University, 420008 Kazan, Russia. favhadiev@ksu.ru

A. Laptev: Department of Mathematics, Imperial College London, London SW7 2AZ, UK, Royal Institute of Technology, 100 44 Stockholm, Sweden. a.laptev@imperial.ac.uk