

EIGENVALUES OF NON-SELFADJOINT FUNCTIONAL DIFFERENCE OPERATORS

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Abstract. Using the well known approach developed in the papers of B.Davies and his co-authors we obtain inequalities for the location of possible complex eigenvalues of non-selfadjoint functional difference operators. When studying the sharpness of the main result we discovered that complex potentials can create resonances.

1. INTRODUCTION

In this paper we are concerned with possible locations of eigenvalues of non-selfadjoint functional difference operators with complex-valued potentials.

Let $P = i\frac{d}{dx}$ be the self-adjoint quantum mechanical momentum operator on $L^2(\mathbb{R})$. For $b > 0$ we consider the Weyl operator $U(b) = \exp(-bP)$. Its domain is given by

$$\text{dom}(U(b)) = \left\{ \psi \in L^2(\mathbb{R}) : e^{-2\pi bk} \widehat{\psi}(k) \in L^2(\mathbb{R}) \right\}$$

with $\widehat{\psi}$ denoting the Fourier transform of ψ ,

$$\widehat{\psi}(k) = (\mathcal{F}\psi)(k) = \int_{\mathbb{R}} e^{-2\pi ikx} \psi(x) dx.$$

Note that functions ψ in the domain $\text{dom}(U(b))$ admit an analytic continuation to $\{z = x + iy \in \mathbb{C} : 0 < y < b\}$ with $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leq y < b$. Furthermore, the limit $\psi(x + ib - i0) = \lim_{\varepsilon \rightarrow 0^+} \psi(x + ib - i\varepsilon)$ exists in the sense of convergence in $L^2(\mathbb{R})$. Denoting this limit by $\psi(x + ib)$, we can write the action of the operator $U(b)$ as

$$(U(b)\psi)(x) = \psi(x + ib).$$

The inverse operator $U^{-1}(b) = \exp(bP)$ and its domain can be characterised similarly.

The functional difference operator $W_0(b)$ is then defined as $W_0(b) = U(b) + U(b)^{-1} = 2 \cosh(bP)$ on the domain

$$\text{dom}(W_0(b)) = \left\{ \psi \in L^2(\mathbb{R}) : 2 \cosh(2\pi bk) \widehat{\psi}(k) \in L^2(\mathbb{R}) \right\}.$$

where it acts as

$$(W_0(\mathfrak{b})\psi)(x) = \psi(x + i\mathfrak{b}) + \psi(x - i\mathfrak{b}).$$

This self-adjoint operator is unitarily equivalent to the multiplication operator $2 \cosh(2\pi\mathfrak{b}k)$. Its spectrum is absolutely continuous and covers the interval $[2, \infty)$ twice.

In this paper our aim is to obtain an estimate for complex eigenvalues of the operator

$$(1.1) \quad W_V(\mathfrak{b}) = W_0(\mathfrak{b}) - V,$$

where V is a complex-valued potential.

In order to describe our result, we first assume that $V \in L^1(\mathbb{R})$ is real-valued. Since $2 \cosh(2\pi\mathfrak{b}k) - 2 > (2\pi\mathfrak{b}k)^2$ it holds that

$$(1.2) \quad W_0(\mathfrak{b}) - 2 > -\mathfrak{b}^2 \frac{d^2}{dx^2}$$

on $\text{dom}(W_0(\mathfrak{b}))$. We can thus apply Sobolev's inequality to conclude that the operator (1.1) is bounded from below on the common domain of $W_0(\mathfrak{b})$ and V . It hence admits a self-adjoint Friedrichs extension, which we continue to denote by $W_V(\mathfrak{b})$. The spectrum of $W_V(\mathfrak{b})$ consists of essential spectrum $[2, \infty)$ and discrete finite-multiplicity eigenvalues below, as can be seen by applying Weyl's theorem and Rellich's lemma together with the observation (see [24, Section 4]) that the form domain of $W_0(\mathfrak{b})$ can be continuously embedded in $H^1(\mathbb{R})$. Details of this argument in the similar case of a Schrödinger operator can be found in the book [14]. As we will recall later, it is also possible to define the operator $W_\nu(\mathfrak{b})$ where the potential V is a finite (real) Borel measure ν .

Any eigenvalue λ of the operator (1.1) with real-valued V can be written as $\lambda = -2 \cos(\omega)$, with $\omega \in [0, \pi)$ for $\lambda \in [-2, 2]$ and $\omega \in i[0, \infty)$ for $\lambda \leq -2$. Under the condition that all eigenvalues $\lambda_j = -2 \cos(\omega_j)$ are larger than or equal to -2 , the authors of [24] proved a Lieb–Thirring type inequality

$$\sum_{j \geq 1} \frac{\sin(\omega_j)}{\omega_j} \leq \frac{1}{2\pi\mathfrak{b}} \int_{\mathbb{R}} |V(x)| dx.$$

As discussed in [24, Remark 1.2], the proof in general does not apply if there is more than one eigenvalue below -2 . However, in the special case that just one of the eigenvalues is below -2 the proof remains applicable. Furthermore, it can also be used to establish that any real eigenvalue $\lambda = -2 \cos(\omega)$, regardless of

whether it lies above or below -2 , must satisfy

$$(1.3) \quad \frac{\sin(\omega)}{\omega} \leq \frac{1}{2\pi b} \int_{\mathbb{R}} |V(x)| dx.$$

The constant $\frac{1}{2\pi b}$ in this inequality is sharp and attained if $V(x) = c\delta(x)$, $c > 0$.

In recent years there has been an increasing interest in eigenvalue estimates for complex-valued potentials. The authors in [1] developed an elegant observation that allows to locate complex eigenvalues for Schrödinger operators with complex-valued potentials. Such an approach and its generalisations were used in [13], [6], [10]. Further development in estimating of complex eigenvalues for Schrödinger operators were obtained in [2], [4], [5], [7], [8], [11], [12], [15], [23], [27] and many others.

It turns out that the inequality (1.3) can be generalised to the non-selfadjoint case. Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$. Denote by

$$(1.4) \quad \Omega = \{\omega \in \mathbb{C} : \operatorname{Re} \omega \in [0, \pi); \operatorname{Im} \omega \in \mathbb{R}\}$$

and

$$(1.5) \quad \Omega_{\pm} = \{\omega \in \mathbb{C} : \operatorname{Re} \omega \in [0, \pi); \operatorname{Im} \omega \in \mathbb{R}_{\pm}\}$$

Then the mapping $\omega \mapsto \lambda(\omega) = -2 \cos(\omega)$ transfers Ω to $\mathbb{C} \setminus [2, \infty)$ and Ω_{\pm} to $\mathbb{C}_{\pm} \setminus [2, \infty)$, where $\mathbb{C}_{\pm} = \{z \in \mathbb{C} : \operatorname{Im} z \in \mathbb{R}_{\pm}\}$.

Our main result is the following.

Theorem 1.1. *Let $V \in L^1(\mathbb{R})$ be a complex-valued potential. Then the eigenvalues $\lambda \in \mathbb{C} \setminus [2, \infty)$ of the operator $W_V(b)$ satisfy the inequality*

$$(1.6) \quad \left| \frac{\sin(\omega)}{\omega} \right| \leq \frac{1}{2\pi b} \int_{\mathbb{R}} |V(x)| dx,$$

where $\lambda = -2 \cos(\omega)$ and where $\omega \in \Omega$.

More generally, let ν be a complex Borel measure of finite total variation $|\nu|(\mathbb{R})$. Then the eigenvalues $\lambda \in \mathbb{C} \setminus [2, \infty)$ of the operator $W_{\nu}(b)$ satisfy the inequality

$$(1.7) \quad \left| \frac{\sin(\omega)}{\omega} \right| \leq \frac{1}{2\pi b} |\nu|(\mathbb{R}).$$

The constant in this inequality is sharp in the sense that there are potentials ν such that inequality (1.7) becomes an equality.

Remark 1.1. *The definition of $W_{\nu}(b)$ as a closed operator will be recalled in the proof of the theorem in Section 4.*

Different aspects of the spectrum of functional difference operators $W_V(b)$ were considered before. In the case when $-V = V_0 = e^{2\pi b x}$, the operator $W_V(b)$ first appeared in the study of the quantum Liouville model on the lattice [9]. It plays an important role in the representation theory of the non-compact quantum group $SL_q(2, \mathbb{R})$. The spectral properties of this operator were studied in [20, 30]. In the case when $-V = 2 \cosh(2\pi b x)$ the operator $W_V(b)$ has entirely discrete spectrum consisting of eigenvalues that diverge to $+\infty$. Weyl laws describing the asymptotics of the eigenvalue counting function and the Riesz means were obtained in [25]. Later, these results were extended to polynomially growing potentials in [26]. Further investigations into the spectral properties of functional difference operators can be found in [16], [17], [21], [22], [29]. The current paper is a part of a programme of the study of different spectral aspects of Weyl operators with decaying potentials. In particular, this programme includes eigenvalue estimates for Dirichlet and Neumann versions of operators $W_V(b)$, trace formulae for (1.1) and their applications to non-linear functional difference operators.

2. RESONANCE STATE

We begin by proving that in the self-adjoint case the spectral point 2 is the resonance state for the operator (1.1).

Theorem 2.1. *Let W_V defined in (1.1) be a self-adjoint, semi-bounded operator such that $V \geq 0$, $V \not\equiv 0$, $V \in L^1(\mathbb{R})$. Then W_V has at least one eigenvalue below the spectral point 2.*

Remark 2.1. *It is well known that for the one-dimensional Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 - V$, $V \geq 0$, $V \not\equiv 0$, there is always at least one negative eigenvalue. Since we have the strict inequality $W_0(b) - 2 > -b^2 \mathrm{d}^2/\mathrm{d}x^2$, Theorem 2.1 cannot be obtained directly from the mentioned result for Schrödinger operators.*

Remark 2.2. *Note that the proof of Theorem 2.1 can be obtained by using Theorem 1.1 from the recent paper [18], where the authors proved a general criterion for zero to be the resonance state. Here we present a direct and simple proof of this fact for $W_V(b)$.*

Proof. For the proof we consider the sequence of test functions

$$u_n(x) = e^{-\frac{x^2}{n^2}} \in \mathrm{dom}(W_V(b)), \quad x \in \mathbb{R}.$$

Clearly for any fixed $x \in \mathbb{R}$ we have $u_n \rightarrow 1$ as $n \rightarrow \infty$. Applying the Fourier transform we obtain

$$\widehat{u}_n(k) = (\mathcal{F}u_n)(k) = \int_{\mathbb{R}} e^{-2\pi i k x} e^{-\frac{x^2}{n^2}} dx = \sqrt{\pi} n e^{-\pi^2 n^2 k^2}$$

and hence

$$\begin{aligned} ((W_V(b) - 2)u_n, u_n) &= \int_{\mathbb{R}} ((W_0(b) - 2)u_n) \overline{u_n} dx - \int_{\mathbb{R}} V|u_n|^2 dx \\ &= \sqrt{\pi} n \int_{\mathbb{R}} (2 \cosh(2\pi b k) - 2) e^{-2\pi^2 n^2 k^2} dk - \int_{\mathbb{R}} V|u_n|^2 dx. \end{aligned}$$

Since

$$n \int_{\mathbb{R}} (2 \cosh(2\pi b k) - 2) e^{-2\pi^2 n^2 k^2} dk \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

we have that there is n_0 such that for any $n > n_0$

$$((W_V(b) - 2)u_n, u_n) < 0.$$

Applying the variational principle we complete the proof. \square

3. FREE RESOLVENT

Since the spectrum $\sigma(W_0(b)) = [2, \infty)$ we conclude that $W_0(b) - \lambda$ is an invertible operator for $\lambda \in \mathbb{C} \setminus [2, \infty)$. Let as before $\lambda = -2 \cos(\omega)$ with $\omega \in \Omega$. Then in Fourier space the inverse of $W_0(b) - \lambda$ is given by the multiplication operator $(2 \cosh(2\pi b k) + 2 \cos(\omega))^{-1}$.

Applying the inverse Fourier transform \mathcal{F}^{-1} to $(2 \cosh(2\pi b k) + 2 \cos(\omega))^{-1}$ we find the kernel of the free resolvent $G_\lambda = (W_0(b) - \lambda)^{-1}$ that is

$$(3.1) \quad G_\lambda(x, y) = G_\lambda(x - y) = \frac{1}{2b \sin(\omega)} \frac{\sinh\left(\frac{\omega}{b}(x - y)\right)}{\sinh\left(\frac{\pi}{b}(x - y)\right)}.$$

In the derivation of this identity using Contour integration, it is essential that $0 \leq \operatorname{Re} \omega < \pi$. If ω had for example been chosen such that $\pi \leq \operatorname{Re} \omega < 2\pi$, the factor ω in (3.1) would have to be replaced by $\omega - 2\pi$, guaranteeing again an exponential decay of the kernel.

Remark 3.1. Note that $G_\lambda(x - y)$ is an even and positive kernel for $\omega \in [0, \pi)$ and it becomes oscillating if $\omega \in i(-\infty, \infty)$.

On the diagonal $x = y$, the kernel G_λ equals

$$(3.2) \quad G_\lambda(0) = \frac{1}{2\pi b} \frac{\omega}{\sin(\omega)}$$

and we can see the relation between the right-hand side of (3.2) and the expression in the left-hand sides of inequalities (1.3) and (1.6). Due to our parameterisation of the spectral parameter, the convergence $\lambda \rightarrow 2$ in $\mathbb{C} \setminus [0, \infty)$ implies $\omega \rightarrow \pi$ in Ω and thus

$$G_\lambda(0) \sim \frac{1}{2b} \frac{1}{\sqrt{1 - \cos^2 \omega}} \sim \frac{1}{2b} \frac{1}{\sqrt{2 - \lambda}}, \quad \text{as } \lambda \rightarrow 2.$$

If $|\lambda| \rightarrow \infty$, then $|\operatorname{Im} \omega| \rightarrow \infty$ and

$$|G_\lambda(0)| \sim \frac{1}{\pi b} |\lambda|^{-1} \log |\lambda|.$$

Proposition 3.1. *For any $\lambda \in \mathbb{C} \setminus [2, \infty)$ we have*

$$(3.3) \quad |G_\lambda(x)| \leq |G_\lambda(0)|, \quad \forall x \in \mathbb{R}.$$

Proof. In order to prove (3.3) it is enough to show

$$\left| \frac{\sinh\left(\frac{\omega}{b}x\right)}{\sinh\left(\frac{\pi}{b}x\right)} \right| \leq \frac{|\omega|}{\pi},$$

where $\omega \in \Omega$ as defined in (1.4). We first prove that for any $\alpha \in \mathbb{C}$ with $0 \leq \operatorname{Re} \alpha \leq 1$ and any $x \in \mathbb{R}$

$$(3.4) \quad |\cosh(\alpha x)| \leq \cosh(x).$$

It suffices to consider $x \geq 0$. We define the holomorphic function $g(\alpha) = \cosh(\alpha x) / \cosh(x)$ on the strip $0 < \operatorname{Re} \alpha < 1$. Clearly it has a continuous extension to $\operatorname{Re} \alpha = 0$ and $\operatorname{Re} \alpha = 1$. On these boundaries it holds that $|g(\alpha)| \leq 1$ since for any $t \in \mathbb{R}$

$$|g(0 + it)| = \frac{|\cosh(itx)|}{\cosh(x)} = \frac{|\cos(tx)|}{\cosh(x)} \leq 1$$

and

$$\begin{aligned} |g(1 + it)|^2 &= \frac{|\cosh(x) \cos(tx) + i \sinh(x) \sin(tx)|^2}{\cosh^2(x)} \\ &= \cos^2(tx) + \tanh^2(x) \sin^2(tx) \leq 1. \end{aligned}$$

On the interior $0 < \operatorname{Re} \alpha < 1$ the function is furthermore bounded

$$|g(\alpha)| = \frac{|e^{\alpha x} + e^{-\alpha x}|}{e^x + e^{-x}} = e^{(\operatorname{Re} \alpha - 1)x} \frac{|1 + e^{-2\alpha x}|}{1 + e^{-2x}} \leq 1 + e^{-2\operatorname{Re} \alpha x} \leq 2.$$

By the Hadamard three-lines theorem (or the Phragmén–Lindelöf principle on vertical strips), we have that $|g(\alpha)| \leq 1$ for all α with $0 \leq \operatorname{Re} \alpha \leq 1$, which proves (3.4).

As a consequence for any such $\alpha \neq 0$ and any $y \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} \left| \frac{\sinh(\alpha y)}{\alpha y} \right| &= \left| \int_0^1 \cosh(\alpha y t) dt \right| \leq \int_0^1 |\cosh(\alpha y t)| dt \\ &\leq \int_0^1 \cosh(y t) dt = \frac{\sinh(y)}{y} = \left| \frac{\sinh(y)}{y} \right|. \end{aligned}$$

Applying this result with $\alpha = \omega/\pi$ and $y = \pi x/b$ we obtain that

$$\left| \frac{\sinh(\frac{\omega}{b} x)}{\omega x} \right| \leq \left| \frac{\sinh(\frac{\pi}{b} x)}{\pi x} \right|$$

for all $\omega \neq 0$ with $0 \leq \operatorname{Re} \omega \leq \pi$ and all $x \in \mathbb{R} \setminus \{0\}$. Rearranging yields the desired result and the proof is complete. \square

Note that in [30] L. Faddeev and L. A. Takhtajan also studied the resolvent. The authors presented it in the form

$$G_\lambda(x-y) = \frac{\sigma}{\sinh(\frac{\pi i \varkappa}{\sigma})} \left(\frac{e^{-2\pi i \varkappa(x-y)}}{1 - e^{-4\pi i \sigma(x-y)}} + \frac{e^{2\pi i \varkappa(x-y)}}{1 - e^{4\pi i \sigma(x-y)}} \right).$$

with $\lambda = 2 \cosh(2b\pi\varkappa)$, $\sigma = i/2b$, $\varkappa = \frac{\omega - \pi}{2\pi i b}$. This expression is equal to (3.1). The authors of [30] pointed out that the resolvent can also be written in a different form. Using generalisations of the Jost solutions

$$f_-(x, \varkappa) = e^{-2\pi i \varkappa x} \quad \text{and} \quad f_+(x, \varkappa) = e^{2\pi i \varkappa x},$$

which are well known in the theory of one-dimensional Schrödinger operators, they established that

$$G_\lambda(x-y) = \frac{2\sigma}{C(f_-, f_+)(\varkappa)} \left(\frac{f_-(x, \varkappa)f_+(y, \varkappa)}{1 - e^{\frac{\pi i}{\sigma'}(x-y)}} + \frac{f_-(y, \varkappa)f_+(x, \varkappa)}{1 - e^{-\frac{\pi i}{\sigma'}(x-y)}} \right).$$

Here $\sigma'\sigma = -1/4$ and

$$C(f, g)(x, \varkappa) = f(x + 2\sigma', \varkappa)g(x, \varkappa) - f(x, \varkappa)g(x + 2\sigma', \varkappa).$$

is the so-called Casorati determinant, which is the analogue of the Wronskian as used in the theory of Schrödinger operators. For the Jost solutions it holds that $C(f_-, f_+)(x, \varkappa) = 2 \sinh(\frac{2\pi \kappa}{\sigma})$ and hence the expression for $G_\lambda(x-y)$ again equals (3.1).

4. PROOF OF THEOREM 1.1

For illustrative purposes we first assume that $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This implies that V is a relatively compact perturbation of $W_0(b)$ and that the operator $W_V(b)$ is closed on the domain of $W_0(b)$, which is continuously embedded in $H^2(\mathbb{R})$. In particular, functions in the domain of $W_V(b)$ are continuous and bounded. We recall the Birman–Schwinger principle [3, 28]. Assuming the existence of an eigenfunction ψ with

$$(4.1) \quad (W_V(b)\psi)(x) = \psi(x + ib) + \psi(x - ib) - V(x)\psi(x) = \lambda\psi(x)$$

we obtain

$$\psi = G_\lambda V\psi.$$

Introducing the multiplication operators

$$(4.2) \quad X = V|V|^{-1/2} \quad \text{and} \quad Y = |V|^{1/2}$$

as well as the function $\varphi = X\psi \in L^2(\mathbb{R})$, we have

$$(4.3) \quad \varphi = XG_\lambda Y\varphi.$$

From (3.1) we find that the integral kernel of the Birman–Schwinger operator $\mathcal{B} = XG_\lambda Y$ equals

$$\mathcal{B}(x, y) = X(x) \frac{1}{2b \sin(\omega)} \frac{\sinh\left(\frac{\omega}{b}(x - y)\right)}{\sinh\left(\frac{\pi}{b}(x - y)\right)} Y(y).$$

The operator \mathcal{B} is furthermore bounded on $L^2(\mathbb{R})$ since by Proposition 3.1

$$\begin{aligned} |(\varphi_1, \mathcal{B}\varphi_2)| &\leq \sup_{x \in \mathbb{R}} |G_\lambda(x)| \|V\|_1 \|\varphi_1\|_2 \|\varphi_2\|_2 \\ &\leq |G_\lambda(0)| \|V\|_1 \|\psi\|_2 \|\varphi\|_2 = \left| \frac{1}{2\pi b} \frac{\omega}{\sin(\omega)} \right| \|V\|_1 \|\psi\|_2 \|\varphi\|_2. \end{aligned}$$

The Birman–Schwinger operator \mathcal{B} has an eigenvalue 1 as established in (4.3), and hence its operator norm is greater or equal to 1. We conclude that

$$\left| \frac{\sin(\omega)}{\omega} \right| \leq \frac{1}{2\pi b} \int_{\mathbb{R}} |V(x)| \, dx.$$

This proves (1.6) for $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

In order to extend the inequality to potentials that are only in $L^1(\mathbb{R})$ and, more generally, to potentials that are complex Borel measures ν of finite total variation, we follow ideas developed in [19]. To this end we have to consider the Birman–Schwinger operator in a different form. It is worth to first elaborate how $W_\nu(b)$ is

defined in this case. Note that $2 \cosh(2\pi bk) = 2 + 4 \sinh(\pi bk)^2$. The quadratic form of $W_0(b)$ is then given by

$$a(\psi) = 2\|\psi\|^2 + \int_{\mathbb{R}} |2 \sinh(\pi bk) \widehat{\psi}(k)|^2 dk$$

and its form domain $\text{dom}(a)$ coincides with the domain of $W_0(b/2)$. For a measure ν of finite total variation we consider the sesquilinear form

$$c(\psi_1, \psi_2) = \int_{\mathbb{R}} \psi_1(x) \overline{\psi_2(x)} d\nu(x).$$

Sobolev's inequality in one dimension states that for all $\psi \in H^1(\mathbb{R})$

$$|\psi(x)|^2 \leq \|\psi'\| \|\psi\|.$$

We conclude that for any $\varepsilon > 0$ and any $\psi \in \text{dom}(a) \subset H^1(\mathbb{R})$

$$\begin{aligned} |c(\psi, \psi)| &\leq |\nu|(\mathbb{R}) \left(\frac{\varepsilon}{2} \int_{\mathbb{R}} |\psi'(x)|^2 dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}} |\psi(x)|^2 dx \right) \\ &\leq |\nu|(\mathbb{R}) \left(\frac{\varepsilon}{2} a(\psi) + \frac{1}{2\varepsilon} \int_{\mathbb{R}} |\psi(x)|^2 dx \right). \end{aligned}$$

Hence the form $a(\psi) + c(\psi, \psi)$ is closed and continuous on $\text{dom}(a)$. For some sufficiently large constant k we also have that

$$a(\psi) + \text{Re } c(\psi, \psi) + k\|\psi\|^2 \geq 0$$

The form thus defines an associated closed operator $W_\nu(b)$.

Let now $\lambda \in \mathbb{C} \setminus [2, \infty)$. We note that by the spectral theorem applied to the self-adjoint operator $W_0(b) \geq 2$, the operators

$$X = \frac{|W_0(b) - \lambda|^{1/2}}{W_0(b) - \lambda} \quad \text{and} \quad Y = |W_0(b) - \lambda|^{-1/2}$$

are well defined and bounded. Their integral kernels are given by

$$\begin{aligned} X(x, z) &= X(x - z) = \mathcal{F}^{-1} \left(\frac{|2 \cosh(2\pi bk) - \lambda|^{1/2}}{2 \cosh(2\pi bk) - \lambda} \right) (x - z), \\ Y(x, z) &= Y(x - z) = \mathcal{F}^{-1} \left(\frac{1}{|2 \cosh(2\pi bk) - \lambda|^{1/2}} \right) (x - z), \end{aligned}$$

and we note that $\overline{Y(x, z)} = Y(x, z)$. We consider the sesquilinear form

$$d(\varphi_1, \varphi_2) = c(X\varphi_1, Y\varphi_2) = \int_{\mathbb{R}} X\varphi_1(x) \overline{Y\varphi_2(x)} d\nu(x).$$

By the Cauchy–Schwarz inequality we have

$$|\mathfrak{d}(\varphi_1, \varphi_2)| \leq \left(\int_{\mathbb{R}} |\mathfrak{X}\varphi_1(x)|^2 \mathfrak{d}|\nu|(x) \right)^{1/2} \left(\int_{\mathbb{R}} |\mathfrak{Y}\varphi_2(x)|^2 \mathfrak{d}|\nu|(x) \right)^{1/2}.$$

Applying the Sobolev inequality, we note that

$$\int_{\mathbb{R}} |\mathfrak{X}\varphi_1(x)|^2 \mathfrak{d}|\nu|(x) \leq |\nu|(\mathbb{R}) \left(\frac{1}{2} \mathfrak{a}(\mathfrak{X}\varphi_1) + \frac{1}{2} \|\mathfrak{X}\varphi_1\|^2 \right)$$

which is smaller than $C\|\varphi_1\|^2$ with a sufficiently large constant C by the spectral theorem. Applying a similar argument to the term with φ_2 , we can conclude that the sesquilinear form \mathfrak{d} is bounded on $L^2(\mathbb{R})$. The Birman–Schwinger operator \mathcal{B}' is then defined as the associated bounded operator and has the integral kernel

$$\mathcal{B}'(x, y) = \int_{\mathbb{R}} \mathfrak{Y}(x, z) \mathfrak{X}(z, y) \mathfrak{d}\nu(z).$$

Formally, the operator is given by $\mathcal{B}' = |W_0(\mathfrak{b}) - \lambda|^{-1/2} \nu \frac{|W_0(\mathfrak{b}) - \lambda|^{1/2}}{W_0(\mathfrak{b}) - \lambda}$.

Assuming now the existence of an eigenfunction ψ of $W_\nu(\mathfrak{b})$, the eigenequation (4.1) implies that

$$\mathfrak{a}(\psi, \psi_2) - \lambda(\psi, \psi_2) = \mathfrak{c}(\psi, \psi_2)$$

for all $\psi_2 \in \text{dom}(\mathfrak{a})$. Since $\text{dom}(\mathfrak{a}) = \text{dom}\left(\frac{W_0(\mathfrak{b}) - \lambda}{|W_0(\mathfrak{b}) - \lambda|^{1/2}}\right) = \text{dom}(|W_0(\mathfrak{b}) - \lambda|^{1/2})$ we can write $\psi = \mathfrak{X}\varphi$ and can replace ψ_2 by $\mathfrak{Y}\varphi_2$ for any $\varphi_2 \in L^2(\mathbb{R})$. We then obtain

$$(\varphi, \varphi_2) = \mathfrak{c}(\mathfrak{X}\varphi, \mathfrak{Y}\varphi_2) = \mathfrak{d}(\varphi, \varphi_2)$$

and conclude that the Birman–Schwinger operator \mathcal{B}' has an eigenvalue 1 and that its operator norm is hence greater or equal to 1. Using $\mathfrak{X} * \mathfrak{Y} = \mathfrak{G}_\lambda$ we can bound the Hilbert–Schmidt norm of \mathcal{B}' as

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{B}'(x, y) \overline{\mathcal{B}'(y, x)} \mathfrak{d}x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathfrak{X}(y - z) \mathfrak{Y}(y - w) \mathfrak{d}y \right) \overline{\left(\int_{\mathbb{R}} \mathfrak{X}(x - w) \mathfrak{Y}(x - z) \mathfrak{d}x \right)} \mathfrak{d}\bar{\nu}(w) \mathfrak{d}\nu(z) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathfrak{G}_\lambda(z - w)|^2 \mathfrak{d}\bar{\nu}(w) \mathfrak{d}\nu(z) \\ &\leq \sup_{x \in \mathbb{R}} |\mathfrak{G}_\lambda(x)|^2 |\nu|(\mathbb{R})^2 = \left| \frac{1}{2\pi\mathfrak{b}} \frac{\omega}{\sin(\omega)} \right|^2 |\nu|(\mathbb{R})^2. \end{aligned}$$

Since the Hilbert–Schmidt norm is larger than the operator norm and the latter is at least 1, we obtain (1.7).

In order to prove that the constant in the inequality (1.7) is sharp we consider the potential $v_c(x) = c\delta(x)$. Here δ is the Dirac δ -function and $c \in \mathbb{C} \setminus [0, \infty)$. The potential v_c is a rank one perturbation of $W_0(b)$ and hence creates at most one eigenvalue outside of the essential spectrum. In Fourier space the eigenequation becomes

$$(4.4) \quad 2 \cosh(2\pi k) \widehat{\psi}_c(k) - c\psi_c(0) = \lambda \widehat{\psi}_c(k).$$

Denoting as before $\lambda = -2 \cos(\omega)$, $\omega \in \Omega$, we obtain

$$\widehat{\psi}_c(k) = \frac{c\psi_c(0)}{2 \cosh(2\pi k) + 2 \cos(\omega)}.$$

Therefore

$$(4.5) \quad \psi_c(x) = c\psi_c(0) G_{-2 \cos(\omega)}(x) = \frac{c\psi_c(0)}{2b \sin(\omega)} \frac{\sinh\left(\frac{\omega}{b}x\right)}{\sinh\left(\frac{\pi}{b}x\right)}.$$

Letting $x \rightarrow 0$ in the last identity we find

$$1 = \frac{c}{2b \sin(\omega)} \frac{\omega}{\pi}$$

and since $|c| = |v_c|(\mathbb{R})$ we conclude that

$$\left| \frac{\sin(\omega)}{\omega} \right| = \frac{1}{2\pi b} |v_c|(\mathbb{R}).$$

The proof of the Theorem 1.1 is complete.

5. EXAMPLES

Let us consider the equation

$$W_0(b)u(x) - c\delta(x)u(x) = \lambda u(x),$$

where $c = re^{i\vartheta}$ with $r > 0$ and $\vartheta \in [0, 2\pi)$. For simplicity we assume that $b = 1$. Then the eigenfunction (4.5) becomes

$$(5.1) \quad \psi_c(x) = \frac{c\psi_c(0)}{2 \sin(\omega)} \frac{\sinh(\omega x)}{\sinh(\pi x)},$$

and ψ_c is in $L^2(\mathbb{R})$ for $\operatorname{Re} \omega \in [0, \pi)$, where it is also an analytic function of ω . However, this function has singularities on the complex line $\omega = \pi + it$, $t \in \mathbb{R}$, and is exponentially growing if $\operatorname{Re} \omega > \pi$. Therefore the equation

$$(5.2) \quad \frac{\sin(\omega)}{\omega} = \frac{r}{2\pi} e^{i\vartheta}$$

defines the eigenvalues $\lambda = -2 \cos(\omega)$ only under the assumption $\operatorname{Re} \omega \in [0, \pi)$. However the equation (5.2) can be solved even for $\operatorname{Re} \omega > \pi$ and thus

gives infinitely many solutions (5.1) to the corresponding eigenequation that are not in $L^2(\mathbb{R})$. It is natural to identify the latter values of λ with resonances. Below we present graphs for three different coupling constants $r/2\pi$, namely $r/2\pi = 2, 0.25$ and 0.2 . We plot the solutions ω of (5.2) for $\vartheta \in [0, 2\pi)$ with

$$\operatorname{Re} \omega \in [0, \pi), \quad \operatorname{Re} \omega \in [\pi, 2\pi) \quad \text{and} \quad \operatorname{Re} \omega \in [2\pi, 3\pi).$$

In each of the plots we highlight the solutions obtained for $\vartheta = \frac{k\pi}{4}$ where $k = 0, \dots, 7$. We also plot the corresponding values $-2 \cos(\omega)$. The complex eigenvalues are given by only the violet curves and the blue and green curves are resonances. In particular, in all three cases we note the absence of a complex eigenvalue if ϑ is sufficiently close to π .

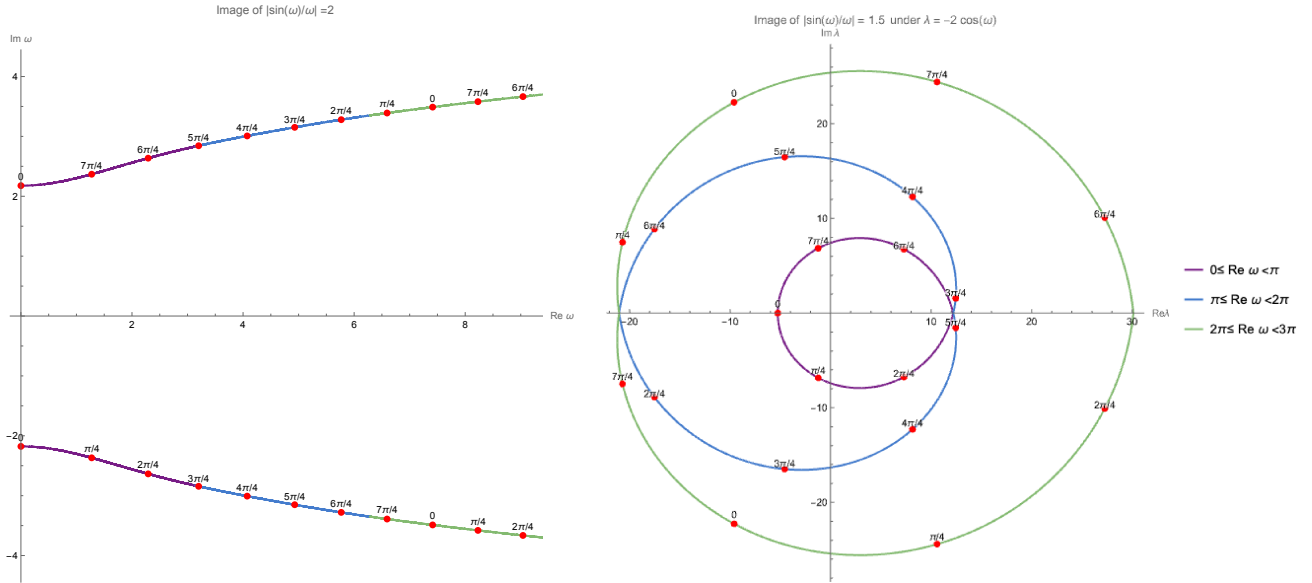


FIGURE 1. The solutions ω and $-2 \cos \omega$ for $r/2\pi = 2$.

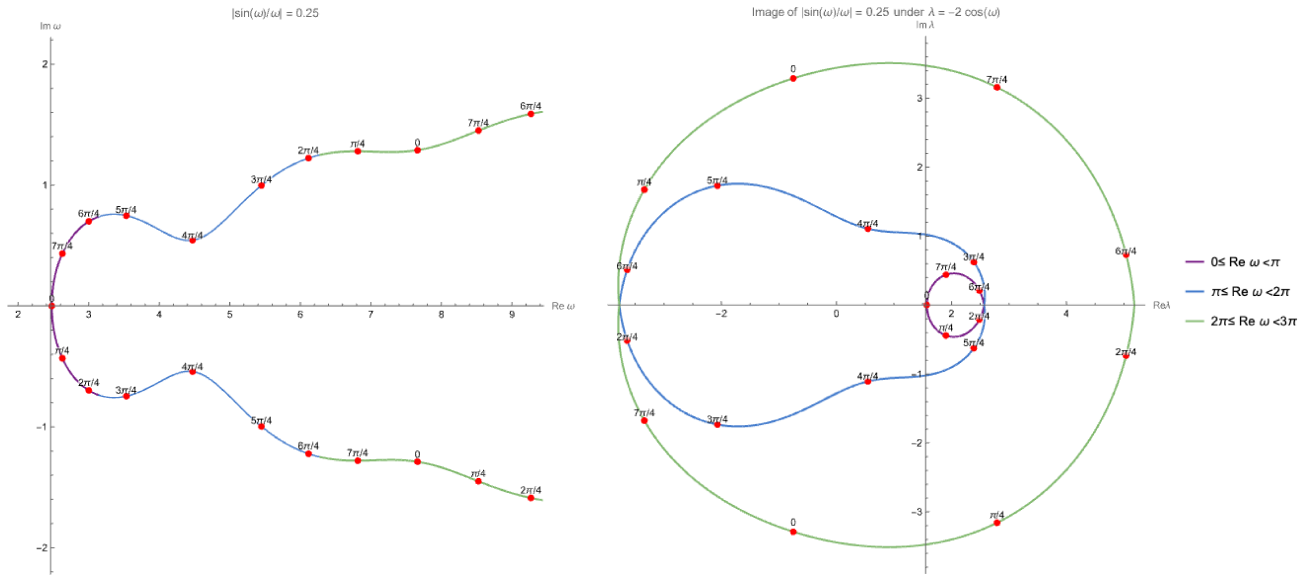


FIGURE 2. The solutions ω and $-2 \cos \omega$ for $r/2\pi = 0.25$.

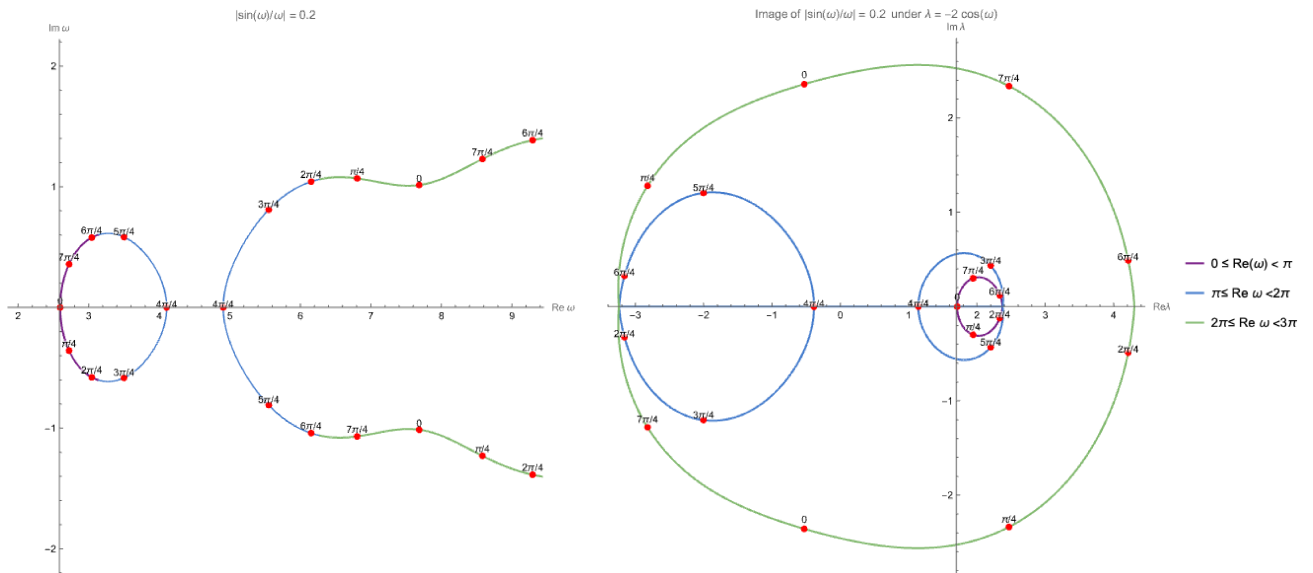


FIGURE 3. The solutions ω and $-2 \cos \omega$ for $r/2\pi = 0.2$.

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