

ON SOME SHARP SPECTRAL INEQUALITIES FOR SCHRÖDINGER OPERATORS ON SEMIAXIS

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ABSTRACT. In this paper we obtain sharp Lieb-Thirring inequalities for a Schrödinger operator on semiaxis with a matrix potential and show how they can be used to other related problems. Among them are spectral inequalities on star graphs and spectral inequalities for Schrödinger operators on half-spaces with Robin boundary conditions.

1. INTRODUCTION

Let us consider a self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V, \quad (1.1)$$

where V is a real-valued function. If the potential function V decays rapidly enough, then the spectrum of the operator H typically is absolutely continuous on $[0, \infty)$. If V has a non-trivial positive part, then H might have finite or infinite number of negative eigenvalues $\{-\lambda_n(H)\}$. If the number of negative eigenvalues is infinite, the point zero is the only possible accumulating point. The inequalities

$$\sum_n \lambda_n^\gamma \leq \frac{R_{\gamma,d}}{(2\pi)^d} \iint_{\mathbb{R}^{2d}} (|\xi|^2 - V(x))_-^\gamma d\xi dx \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} dx \quad (1.2)$$

are known as Lieb-Thirring bounds. Here and in the following, $V_\pm = (|V| \pm V)/2$ denote the positive and negative parts of the function V .

It is known that the inequality (1.2) holds true with some finite constants if and only if $\gamma \geq 1/2$, $d = 1$; $\gamma > 0$, $d = 2$ and $\gamma \geq 0$, $d \geq 3$. There are examples showing that (1.2) fails for $0 \leq \gamma < 1/2$, $d = 1$ and $\gamma = 0$, $d = 2$.

Almost all the cases except for $\gamma = 1/2$, $d = 1$ and $\gamma = 0$, $d \geq 3$ were justified in the original paper of E.H.Lieb and W.Thirring [LT]. The critical case $\gamma = 0$, $d \geq 3$ is known as the Cwikel-Lieb-Rozenblum inequality, see [Cw, L, Roz]. It was also proved in [Fe, LY, Con] and very recently by R. Frank [Fr] using Rumin's approach. The remaining case $\gamma = 1/2$, $d = 1$ was verified by T.Weidl in [W1].

The sharp value of the constants $R_{\gamma,d} = 1$ in (1.2) are known for the case $\gamma \geq 3/2$ in all dimensions and it was first proved in [LT] and [AizL] for $d = 1$ and later in

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[LW1, LW2] for any dimension. In this case

$$L_{\gamma,d} = L_{\gamma,d}^{cl} := (2\pi)^{-d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\gamma d\xi.$$

The only other case where the sharp value of the constant $R_{\gamma,d}$ is known is the case $R_{1/2,1} = 2$. This result was obtained by D.Hundertmark, E.H.Lieb and L.E.Thomas in [HLT].

In this paper we consider a one-dimensional systems of Schrödinger operators acting in $L^2(\mathbb{R}_+, \mathbb{C}^N)$, $\mathbb{R}_+ = (0, \infty)$, defined by

$$\mathcal{H}\varphi(x) = \left(-\frac{d^2}{dx^2} \otimes \mathbb{I} - V(x) \right) \varphi(x), \quad \varphi'(0) - \mathfrak{S} \varphi(0) = 0, \quad (1.3)$$

where \mathbb{I} is the $N \times N$ identity matrix, V is a Hermitian $N \times N$ matrix-function and \mathfrak{S} is a $N \times N$ Hermitian matrix.

Assuming that the potential V generates only a discrete negative spectrum, we denote by $\{-\lambda_n\}$ the negative eigenvalues of \mathcal{H} .

One of the main results of this paper is the following:

Theorem 1.1. *Let $\text{Tr } V^2 \in L^1(\mathbb{R}_+)$, $V \geq 0$. Then the negative spectrum of the operator \mathcal{H} defined in (1.3) is discrete and the following Lieb-Thirring inequality for its eigenvalues $\{-\lambda_n\}$ holds*

$$\begin{aligned} \frac{3}{4} \lambda_1 \text{Tr } \mathfrak{S} + \frac{1}{2} (2\kappa_1 - N) \lambda_1^{3/2} + \sum_{n=2}^{\infty} \kappa_n \lambda_n^{3/2} \\ \leq \frac{3}{16} \int_0^\infty \text{Tr } V^2(x) dx + \frac{1}{4} \text{Tr } \mathfrak{S}^3, \quad (1.4) \end{aligned}$$

where κ_n is the multiplicity of the eigenvalue $-\lambda_n$.

Examples.

1. *Let $V \equiv 0$ and $N = 1$. Then the boundary value problem*

$$-\varphi''(x) = -\lambda\varphi(x), \quad \varphi'(0) - \sigma\varphi(0) = 0, \quad \sigma < 0,$$

has only one L^2 -solution

$$\varphi(x) = C e^{-\sqrt{\lambda}x}, \quad -\sqrt{\lambda} = \sigma.$$

In this case the inequality (1.4) becomes saturated, $\frac{3}{4} \sigma^3 - \frac{1}{2} \sigma^3 \leq \frac{1}{4} \sigma^3$.

2. *Let $N = 2$, $V \equiv 0$ and*

$$\mathfrak{S} = \begin{pmatrix} \sigma & 0 \\ 0 & -\alpha\sigma \end{pmatrix}, \quad \sigma < 0$$

2a) If $\alpha \geq 0$ then the boundary value problem (1.3) has one negative eigenvalue $-\lambda$ of multiplicity one satisfying the identity $-\sqrt{\lambda} = \sigma$. In this case $2\kappa_1 - N = 0$ and the inequality (1.4) becomes

$$3\lambda \operatorname{Tr} \mathfrak{S} = 3\lambda \sigma (1 - \alpha) \leq (1 - \alpha^3) \sigma^3 = \operatorname{Tr} \mathfrak{S}^3,$$

or

$$3(\alpha - 1) \leq \alpha^3 - 1.$$

which holds true for any $\alpha \geq 0$.

2b) If $-1 < \alpha < 0$, then the problem (1.3) has two eigenvalues satisfying $-\sqrt{\lambda_1} = \sigma$ and $-\sqrt{\lambda_2} = -\alpha\sigma$ and (1.4) is reduced to

$$3(\alpha - 1) - 4\alpha^3 \leq \alpha^3 - 1.$$

2c) Finally, if $\alpha = -1$, then $-\sqrt{\lambda_1} = \sigma$ is of multiplicity $\kappa_1 = 2$ and (1.4) becomes an identity.

Note that if $\operatorname{Tr} \mathfrak{S}^3 \leq 0$, then the inequality (1.4) implies

$$\frac{3}{4} \lambda_1 \operatorname{Tr} \mathfrak{S} + \frac{1}{2} (2\kappa_1 - N) \lambda_1^{3/2} + \sum_{n=2}^{\infty} \kappa_n \lambda_n^{3/2} \leq \frac{3}{16} \int_0^{\infty} \operatorname{Tr} V^2(x) dx. \quad (1.5)$$

The latter allows us to use the standard Aizenman-Lieb arguments [AizL] and to derive

Corollary 1.2. Let $\operatorname{Tr} \mathfrak{S}^3 \leq 0$, $V \geq 0$ and $\operatorname{Tr} V^{\gamma+1/2}(x) \in L^1(0, \infty)$. Then for any $\gamma \geq 3/2$ we have

$$\begin{aligned} \frac{\mathcal{B}(\gamma - 3/2, 2)}{\mathcal{B}(\gamma - 3/2, 5/2)} \frac{3}{4} \lambda_1^{\gamma-1/2} \operatorname{Tr} \mathfrak{S} + \frac{1}{2} (2\kappa_1 - N) \lambda_1^{\gamma} + \sum_{n=2}^{\infty} \kappa_n \lambda_n^{\gamma} \\ \leq L_{\gamma,1}^{cl} \int_0^{\infty} \operatorname{Tr} (V(x))^{\gamma+1/2} dx, \end{aligned}$$

where by $\mathcal{B}(p, q)$ we denote the classical Beta function

$$\mathcal{B}(p, q) = \int_0^1 (1-t)^{q-1} t^{p-1} dt.$$

Corollary 1.3. If $\mathfrak{S} = 0$, then (1.3) can be identified with the Neumann boundary value problem and we obtain

$$\frac{1}{2} (2\kappa_1 - N) \kappa_1 \lambda_1^{\gamma} + \sum_{n=2}^{\infty} \kappa_n \lambda_n^{\gamma} \leq L_{\gamma,1}^{cl} \int_0^{\infty} \operatorname{Tr} (V(x))^{\gamma+1/2} dx, \quad \gamma \geq 3/2.$$

Remark.

Note that in the scalar case $N = 1$ we obtain

$$\frac{1}{2} \lambda_1^\gamma + \sum_{n=2}^{\infty} \lambda_n^\gamma \leq L_{\gamma,1}^{cl} \int_0^\infty V^{\gamma+1/2}(x) dx, \quad \gamma \geq 3/2, \quad (1.6)$$

which means that the semi-classical inequality holds true for all eigenvalues starting from $n = 2$ and that in the latter inequality the Neumann boundary condition affects only the first eigenvalue.

If $V \geq 0$ is a diagonal $N \times N$ matrix-function, then the operator \mathcal{H} could be interpreted as a Schrödinger operator on a star graph with N edges; the matrix \mathfrak{S} describes a vertex coupling without the Dirichlet component [Ku]. In such a case we obtain:

Theorem 1.4. *Let $V \geq 0$ be a diagonal $N \times N$ matrix-function and let \mathfrak{S} be a Hermitian matrix. Then the operator (1.3) can be identified with a Schrödinger operator on a star graph with N semi-infinite edges and its negative spectrum satisfies the inequality (1.4).*

If both $V \geq 0$ and \mathfrak{S} are diagonal $N \times N$ matrices, then the negative spectrum of the operator \mathcal{H} is the union of the eigenvalues from each channel and we obtain

Theorem 1.5. *Let $V \geq 0$, $\text{Tr } V^2 \in L^1(0, \infty)$ and let V and \mathfrak{S} be diagonal $N \times N$ matrices with entries v_j and σ_j , $j = 1, \dots, N$, respectively. Then the negative eigenvalues of the operator \mathcal{H} defined in (1.3), satisfy the inequality*

$$\frac{3}{4} \sum_{j=1}^N \lambda_{j1} \sigma_j + \frac{1}{2} \sum_{j=1}^N \lambda_{j1}^{3/2} + \sum_{j=1}^N \sum_{n=2}^{\infty} \lambda_{jn}^{3/2} \leq \frac{3}{16} \int_0^\infty \text{Tr } V^2(x) dx + \frac{1}{4} \text{Tr } \mathfrak{S}^3, \quad (1.7)$$

where $-\lambda_{jn}$ are negative eigenvalues of operators h_j defined by

$$h_j \psi(x) = -\frac{d^2}{dx^2} \psi(x) - v_j(x) \psi(x), \quad \psi'(0) - \sigma_j \psi(0) = 0.$$

Remark. *Note that the inequality (1.7) is much more precise than (1.4) due to the diagonal structure of the operator \mathcal{H} . In (1.7) all N first eigenvalues generated by each channel are affected by the Robin boundary conditions, whereas in (1.4) only the first one, see Example 2b).*

We now give an example how our results could be applied for spectral estimates of multi-dimensional Schrödinger operators.

Let $\mathbb{R}_+^d = \{x = (x_1, x') : x_1 > 0, x' \in \mathbb{R}^{d-1}\}$ and let H be a Schrödinger operator in $L^2(\mathbb{R}_+^d)$ with the Neumann boundary conditions

$$H\psi = -\Delta\psi - V\psi = -\lambda\psi, \quad \frac{\partial}{\partial x_1} \psi(0, x') = 0. \quad (1.8)$$

The following result can be obtained by a ‘‘lifting’’ argument with respect to dimension, see [L], [LT]:

Theorem 1.6. *Let $V \geq 0$ and $V \in L^{\gamma+d/2}$, $\gamma \geq 3/2$. Then for the negative eigenvalues $\{-\lambda_n\}$ of the operator (1.8) we have*

$$\begin{aligned} \sum_n \lambda_n^\gamma &\leq L_{\gamma,d}^{cl} \int_{\mathbb{R}_+^d} V^{\gamma+d/2}(x) dx + \frac{1}{2} L_{\gamma,d-1}^{cl} \int_{\mathbb{R}^{d-1}} \mu_1^{\gamma+(d-1)/2}(x') dx' \\ &\leq 2 L_{\gamma,d}^{cl} \int_{\mathbb{R}_+^d} V^{\gamma+d/2} dx. \end{aligned} \quad (1.9)$$

Here $\mu_1(x')$ is the ground state energy for the operator $-d^2/dx_1^2 - V(x_1, x')$ in $L^2(\mathbb{R}_+)$ with the Neumann boundary condition at zero.

Remark. *Note that the inequality $\sum_n \lambda_n^\gamma \leq 2 L_{\gamma,d}^{cl} \int_{\mathbb{R}_+^d} V^{\gamma+d/2} dx$ can be obtained directly by extending the operator (1.8) to the whole space $L^2(\mathbb{R}^d)$ with the symmetrically reflected potential.*

Remark. *Note also that in the paper [FL] some sharp spectral inequalities for Schrödinger operators with δ -type potentials concentrated on hyperplanes were derived.*

Finally we would like to mention that in the paper [BT] the authors obtained a Schmincke type inequality for a scalar Schrödinger operator on the semiaxes with Robin boundary conditions, that is an estimate from below for the value $\sum_{n=1}^{\infty} \sqrt{\lambda_n}$.

2. SOME AUXILIARY RESULTS

In this Section we assume that the matrix-function V is compactly supported, $\text{supp } V \subset [a, b]$ for some $a, b : 0 < a < b < \infty$ and adapt the arguments from [BL] to the case of semiaxis.

We begin with stating a well-known fact concerning the ground state of the operator (1.3).

Lemma 2.1. *Let $-\lambda < 0$ be the ground state energy of the operator \mathcal{H} and let $\varphi(x) = \{\varphi_k\}_{k=0}^N$ be a $L^2(\mathbb{R}_+, \mathbb{C}^N)$ -vector-function satisfying the equation*

$$\mathcal{H}\varphi(x) = -\frac{d^2}{dx^2}\varphi(x) - V(x)\varphi(x) = -\lambda\varphi(x), \quad \varphi'(0) - \mathfrak{S}\varphi(0) = 0, \quad (2.1)$$

and such that the $2N$ vector $(\varphi(0), \varphi'(0))$ is not trivial. Then $\varphi(x) \neq 0$, $x \in \mathbb{R}_+$, and the ground state energy multiplicity is at most N .

Proof. Suppose that $\varphi(x_0) = 0$ for some $x_0 > 0$. Consider the continuous function

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & x < x_0 \\ 0, & x \geq x_0. \end{cases}$$

This function is non-trivial, belongs to the Sobolev space $H^1(\mathbb{R}_+, \mathbb{C}^N)$ and satisfies

$$\begin{aligned} \int_{\mathbb{R}_+} (|\tilde{\varphi}'|^2 - (V\tilde{\varphi}, \tilde{\varphi})_{\mathbb{C}^N}) dx &= \int_0^{x_0} (|\varphi'|^2 - (V\varphi, \varphi)_{\mathbb{C}^N}) dx \\ &= \int_0^{x_0} (-\varphi'' - V\varphi, \varphi)_{\mathbb{C}^N} dx = -\lambda \int_0^{x_0} |\varphi|^2 dx = -\lambda \int_{\mathbb{R}_+} |\tilde{\varphi}|^2 dx. \end{aligned}$$

Therefore $\tilde{\varphi}$ minimizes the closed quadratic form associated with \mathcal{H} . Thus by the variational principle $\tilde{\varphi}$ belongs to the domain of \mathcal{H} and solves the Cauchy problem pointwise. However, since $\tilde{\varphi}(x) = 0$ for $x \geq x_0$ it also solves the backward Cauchy problem with zero initial data at x_0 and by uniqueness must vanish everywhere. This contradicts the non-triviality of $\tilde{\varphi}$ for $x < x_0$. \square

Similarly to [BL] let us introduce a (not necessary L^2) fundamental $N \times N$ -matrix-solution $M(x)$ of the equation (2.1), where $-\lambda$ is the ground state energy for the operator \mathcal{H} , so M satisfies the equation

$$-\frac{d^2}{dx^2}M(x) - V(x)M(x) = -\lambda M(x), \quad M'(0) - \mathfrak{S}M(0) = 0. \quad (2.2)$$

Denoting $M(0) = A$ and $M'(0) = B$, $B - \mathfrak{S}A = 0$, we shall always assume that the matrix A is invertible.

By using Lemma 2.1 we obtain that the matrix-function $M(x)$ is invertible for any $x \in \mathbb{R}_+$ and thus we can consider

$$F(x) = M'(x)M^{-1}(x). \quad (2.3)$$

Lemma 2.2. *The matrix function $F(x)$ satisfies the following properties:*

- $F(x)$ is Hermitian for any $x \in \mathbb{R}_+$.
- $F(x)$ is independent of the choice of the matrices A, B , satisfying the equation $B - \mathfrak{S}A = 0$ and

$$F(0) = B A^{-1} = \mathfrak{S}.$$

- F satisfies the matrix Riccati equation

$$F'(x) + F^2(x) + V(x) = \lambda \mathbb{I}. \quad (2.4)$$

Proof. From the Wronskian identity

$$\frac{d}{dx}W(x) := \frac{d}{dx} \left(M^*(x)M'(x) - (M^*(x))'M(x) \right) = 0$$

we obtain

$$W(x) = M^*(x)M'(x) - (M^*(x))'M(x) = \text{const.}$$

Since $M(0) = A$ and $M'(0) = B$, using the fact that \mathfrak{S} is Hermitian we find

$$\begin{aligned} W(0) &= M^*(0) M'(0) - (M^*(0))' M(0) \\ &= A^* \left(B A^{-1} - (A^*)^{-1} B^* \right) A = A^* (\mathfrak{S} - \mathfrak{S}^*) A = 0. \end{aligned}$$

Thus

$$W(x) = M^*(x) M'(x) - (M^*(x))' M(x) = 0.$$

Multiplying the latter identity by M^{-1} from the right and by $(M^{-1})^*$ from the left we obtain $F(x) = F^*(x)$. Moreover

$$\begin{aligned} F' + F^2 &= (M' M^{-1})' + (M' M^{-1})^2 \\ &= M'' M^{-1} - M' M^{-1} M' M^{-1} + M' M^{-1} M' M^{-1} = (\lambda - V) M M^{-1} = \lambda \mathbb{I} - V. \end{aligned}$$

□

Next, we analyze the behavior of the matrices $F(x)$ and their eigenvalues and eigenvectors as $x \rightarrow \infty$. For $x > b$ any solution of the differential equation (2.2) can be written as

$$\begin{aligned} M(x) &= \cosh(\sqrt{\lambda}(x-b))M(b) + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x-b))M'(b) \\ &= \left(\cosh(\sqrt{\lambda}(x-b))\mathbb{I} + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x-b))F(b) \right) M(b). \end{aligned} \quad (2.5)$$

With the help of this representation we show

Lemma 2.3. *For all $x \geq b$ it holds $F(x) = f(x, F(b))$, where*

$$f(x, \mu) = \sqrt{\lambda} \frac{\sqrt{\lambda} \tanh(\sqrt{\lambda}(x-b)) + \mu}{\sqrt{\lambda} + \mu \tanh(\sqrt{\lambda}(x-b))}. \quad (2.6)$$

Proof. In view of (2.5) we have

$$M'(x) = \left(\sqrt{\lambda} \sinh(\sqrt{\lambda}(x-b))\mathbb{I} + \cosh(\sqrt{\lambda}(x-b))F(b) \right) M(b),$$

$$(M(x))^{-1} = (M(b))^{-1} \left(\cosh(\sqrt{\lambda}(x-b))\mathbb{I} + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x-b))F(b) \right)^{-1}.$$

It remains to insert these expressions in the definition $F(x) = M'(x)(M(x))^{-1}$ and to apply the spectral theorem for the Hermitian matrix $F(b)$. □

Note that $f(x, \mu)$ is strictly monotone in μ . As a direct consequence of Lemma 2.3 we conclude, that the eigenvectors of the matrix $F(x)$ are independent of x for $x \geq b$ as vectors in \mathbb{C}^N . Moreover, the eigenvalues of F may or may not depend on x outside the support of V depending on if they correspond to growing or decaying solutions.

Corollary 2.4. *Each eigenvalue μ_k of $F(b)$ gives rise to a continuous eigenvalue branch $\mu_k(x) = f(x, \mu_k(b))$. In particular, we have*

$$\mu_k(x) = -\sqrt{\lambda} \quad \text{iff} \quad \mu_k(b) = -\sqrt{\lambda},$$

and

$$\lim_{x \rightarrow \infty} \mu_k(x) = \sqrt{\lambda} \quad \text{iff} \quad \mu_k(b) \neq -\sqrt{\lambda}.$$

The limit in the last expression is achieved exponentially fast.

Remark. *There is a one-to-one correspondence between the \varkappa_1 -dimensional space of ground states for \mathcal{H} and a \varkappa_1 -dimensional eigenspace of $F(b)$ corresponding to the eigenvalue $-\sqrt{\lambda}$. Indeed, since $M(x)$ is a fundamental system of the solutions of the Cauchy problem (2.1) and $F(b)$ is invertible, any particular solution φ of (2.1) can be represented as $\varphi(x) = F(x)(F(b))^{-1} \nu$ with some $\nu \in \mathbb{C}^N$. Hence, by (2.5)*

$$\begin{aligned} \varphi_\nu(x) &= \cosh(\sqrt{\lambda}(x-b)) \nu + \frac{1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(x-b)) F(b) \nu \\ &= \frac{1}{2\sqrt{\lambda}} e^{\sqrt{\lambda}(x-b)} \left(\sqrt{\lambda} \nu + F(b) \nu \right) - \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(x-b)} \left(\sqrt{\lambda} \nu - F(b) \nu \right). \end{aligned} \quad (2.7)$$

This function becomes an L^2 -eigenfunction of \mathcal{H} , if and only if $F(b) \nu = -\sqrt{\lambda} \nu$.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1 .

Let now $-\lambda_1$ be the ground state energy of the operator \mathcal{H} with multiplicity $\varkappa_1 \leq N$, let $M_1(x)$ be a fundamental system of solutions corresponding the eigenvalue $-\lambda_1$ and $F_1 = M_1' M_1^{-1}$. We consider the operator

$$Q_1 = \frac{d}{dx} \otimes \mathbb{I} - F_1(x)$$

and its adjoint

$$Q_1^* = -\frac{d}{dx} \otimes \mathbb{I} - F_1(x)$$

in $L^2(\mathbb{R}^+, \mathbb{C}^N)$, assuming that they are closed operators on $H^1(\mathbb{R}^+, \mathbb{C}^N)$. Using Riccati's equation (2.4) we obtain the following factorization of the original operator \mathcal{H}

$$Q_1^* Q_1 = -\frac{d^2}{dx^2} \otimes \mathbb{I} + F_1'(x) + (F_1(x))^2 = \mathcal{H} + \lambda_1 \mathbb{I}.$$

Consider

$$Q_1 Q_1^* = -\frac{d^2}{dx^2} \otimes \mathbb{I} - V(x) - 2F_1'(x) + \lambda_1 \mathbb{I} = \mathcal{H} - 2F_1'(x) + \lambda_1 \mathbb{I}.$$

Note that non-zero eigenvalues of $Q_1^* Q_1$ and $Q_1 Q_1^*$ are same. However, while the vector-eigenfunctions φ defined in (2.7) satisfy the boundary conditions

$$\varphi'(0) - \mathfrak{S} \varphi(0) = 0,$$

the vector-eigenfunctions of $Q_1 Q_1^*$ satisfy the Dirichlet boundary condition at 0.

Indeed, if φ is a vector-eigenfunction of $Q_1^* Q_1$ satisfying $\varphi'(0) - \mathfrak{S}\varphi(0) = 0$ then $\psi = Q_1 \varphi$ is an eigenfunction of $Q_1 Q_1^*$ and

$$\psi(0) = (Q_1 \varphi)(0) = \varphi'(0) - F_1(0)\varphi(0) = 0.$$

Next, let us verify that the kernel $\ker Q_1^*$ is trivial, and consequently, $0 \notin \text{spec}(Q_1 Q_1^*)$. Indeed, assume for a moment that there is a non-trivial vector-function ψ satisfying the Dirichlet boundary conditions at $x = 0$ and such that

$$Q_1 Q_1^* \psi = 0. \quad (3.1)$$

Then

$$(Q_1 Q_1^* \psi, \psi) = \|Q_1^* \psi\|^2 = 0.$$

However, $Q_1^* \psi = 0$ if and only if $\psi'(x) = F(x)\psi(x)$ for all $x \in \mathbb{R}_+$ and, in particular, $\psi'(0) = F(0)\psi(0) = 0$. Since ψ satisfies the equation (3.1) together with $\psi(0) = \psi'(0) = 0$ we obtain that $\psi \equiv 0$.

Hence, the negative spectra of \mathcal{H} and $\mathcal{H} - 2F_1'$ coincide except for the spectral value of the ground state energy, which does not belong to the spectrum of $\mathcal{H} - 2F_1'$ anymore. We emphasize that this commutation method removes even a \varkappa_1 -fold degenerate ground state energy $-\lambda_1$ of \mathcal{H} .

Therefore the spectral problem for the operator (1.3) is reduced to the operator in $L^2(\mathbb{R}_+)$

$$\mathcal{H}_1 \psi = \left(-\frac{d^2}{dx^2} \otimes I - V(x) - 2F_1' \right) \psi = -\lambda \psi \quad \psi(0) = 0.$$

Let us extend V by zero to the negative semi-axis. Using then the variational principle we can apply the well-known Lieb-Thirring inequalities for 1D Schrödinger operators with matrix-valued potentials (see [LW1], [BL]) and obtain

$$\begin{aligned} \sum_{n=2}^{\infty} \varkappa_n \lambda_n^{3/2} &\leq \frac{3}{16} \int_0^{\infty} \text{Tr} (V(x) + 2F_1'(x))^2 dx \\ &= \frac{3}{16} \int_0^{\infty} \text{Tr} \left(V^2(x) + 4F_1'(x)(V(x) + F_1'(x)) \right) dx. \end{aligned}$$

Using the Riccati equation (2.4), the fact that the matrix $\lim_{x \rightarrow \infty} F(x)$ has the eigenvalue $-\sqrt{\lambda_1}$ of multiplicity \varkappa_1 and the eigenvalue $\sqrt{\lambda_1}$ of multiplicity $N - \varkappa_1$ and

that $F(0) = \mathfrak{S}$, we finally arrive at

$$\begin{aligned}
\sum_{n=2}^{\infty} \varkappa_n \lambda_n^{3/2} &\leq \frac{3}{16} \int_0^{\infty} \operatorname{Tr} \left(V^2(x) + 4F_1'(x)(\lambda_1 - F_1^2(x)) \right) dx \\
&= \frac{3}{16} \int_0^{\infty} \operatorname{Tr} V^2(x) dx + \frac{3}{4} \lambda_1 \operatorname{Tr} F_1(x) \Big|_0^{\infty} - \frac{1}{4} \operatorname{Tr} F_1^3(x) \Big|_0^{\infty} \\
&= \frac{3}{16} \int_0^{\infty} \operatorname{Tr} V^2(x) dx + \frac{3}{4} \lambda_1 \left(-\varkappa_1 \sqrt{\lambda_1} + (N - \varkappa_1) \sqrt{\lambda_1} - \operatorname{Tr} \mathfrak{S} \right) \\
&\quad - \frac{1}{4} \left(-\varkappa_1 \lambda_1^{3/2} + (N - \varkappa_1) \lambda_1^{3/2} - \operatorname{Tr} \mathfrak{S}^3 \right) \\
&= \frac{3}{16} \int_0^{\infty} \operatorname{Tr} V^2(x) dx - \frac{1}{2} (2\varkappa_1 - N) \lambda_1^{3/2} - \frac{3}{4} \lambda_1 \operatorname{Tr} \mathfrak{S} + \frac{1}{4} \operatorname{Tr} \mathfrak{S}^3.
\end{aligned}$$

Finally using standard arguments we can consider the closure of the latter inequality from the class of compactly supported potentials to the class $L^2(\mathbb{R}_+, \mathbb{C}^N \times \mathbb{C}^N)$. The proof of Theorem 1.1 is complete.

Proof of Corollary 1.2.

Let us denote by $\lambda_n = \lambda_n(V)$ the eigenvalues of the Schrödinger operator with the potential V . Then by using the variational principle and the inequality (1.5) we find that for any $\gamma > 3/2$

$$\begin{aligned}
&\mathcal{B}(\gamma - 3/2, 2) \frac{3}{4} \operatorname{Tr} \mathfrak{S} \lambda_1^{\gamma-1/2}(V) \\
&\quad + \mathcal{B}(\gamma - 3/2, 5/2) \left(\frac{1}{2} (2\varkappa_1 - N) \lambda_1^{\gamma}(V) + \sum_{n=2}^{\infty} \varkappa_n \lambda_n^{\gamma}(V) \right) \\
&\quad = \int_0^{\infty} \left(\frac{3}{4} \operatorname{Tr} \mathfrak{S} (\lambda_1(V) - t)_+ \right. \\
&\quad + \frac{1}{2} (2\varkappa_1 - N) (\lambda_1(V) - t)_+^{3/2} + \sum_{n=2}^{\infty} \varkappa_n (\lambda_n(V) - t)_+^{3/2} \Big) t^{\gamma-5/2} dt \\
&\quad \leq \int_0^{\infty} \left(\frac{3}{4} \operatorname{Tr} \mathfrak{S} (\lambda_1((V - t)_+)) \right. \\
&\quad + \frac{1}{2} (2\varkappa_1 - N) (\lambda_1((V - t)_+))^{3/2} + \sum_{n=2}^{\infty} \varkappa_n (\lambda_n((V - t)_+))^{3/2} \Big) t^{\gamma-5/2} dt \\
&\quad \leq \frac{3}{16} \int_0^{\infty} \int_0^{\infty} \operatorname{Tr} (V(x) - t)_+^2 t^{\gamma-5/2} dt dx \\
&\quad = \mathcal{B}(\gamma - 3/2, 3) \frac{3}{16} \int_0^{\infty} \operatorname{Tr} V^{\gamma+1/2}(x) dx.
\end{aligned}$$

Dividing by $\mathcal{B}(\gamma - 3/2, 5/2)$ and noting that

$$\frac{3}{16} \frac{\mathcal{B}(\gamma - 3/2, 3)}{\mathcal{B}(\gamma - 3/2, 5/2)} = L_{\gamma+1/2, 1}^{cl}$$

we complete the proof.

Proof of Theorem 1.6.

Let $\{\mu_j(x')\}$ be eigenvalues of the Neumann problem for the Schrödinger operator

$$-\frac{d^2}{dx_1^2}\psi(x_1, x') - V(x_1, x')\psi(x_1, x') = -\mu(x')\psi(x_1, x')$$

considering x' as a parameter.

For any $\gamma \geq 3/2$ and $d \geq 1$ let us apply the operator version of the Lieb-Thirring inequality (see [LW1]) with respect to \mathbb{R}^{d-1} and obtain

$$\sum_n \lambda_n^\gamma \leq L_{\gamma, d-1}^{cl} \int_{\mathbb{R}^{d-1}} \sum_j \mu_j^{\gamma+(d-1)/2}(x') dx'.$$

By using (1.6) we find

$$\begin{aligned} \sum_j \mu_j^{\gamma+(d-1)/2}(x') &\leq \frac{1}{2} \mu_1^{\gamma+(d-1)/2}(x') + L_{\gamma+(d-1)/2, 1}^{cl} \int_0^\infty V^{\gamma+d/2}(x_1, x') dx_1 \\ &\leq 2 L_{\gamma+(d-1)/2, 1}^{cl} \int_0^\infty V^{\gamma+d/2}(x_1, x') dx_1. \end{aligned}$$

Noticing that

$$L_{\gamma, d-1}^{cl} L_{\gamma+(d-1)/2, 1}^{cl} = L_{\gamma, d}^{cl}$$

we obtain the proof.

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REFERENCES

- [AizL] M. Aizenman and E.H. Lieb, *On semi-classical bounds for eigenvalues of Schrödinger operators*, Phys. Lett. **66A** (1978), 427–429.
- [BT] A. Boumenir and V. K. Tuan, *A trace formula and Schmincke inequality on the half-line*, Proc. Amer. Math. Soc., **137** (3) (2009) 1039–1049.
- [BL] R. Benguria and M. Loss, *A simple proof of a theorem by Laptev and Weidl*, Math. Res. Lett. **7** (2000), no. 2-3, 195–203.
- [Con] J.G. Conlon, *A new proof of the Cwikel-Lieb-Rosenbljum bound*, Rocky Mountain J. Math., **15** (1985), 117–122.
- [Cw] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Trans. AMS, **224** (1977), 93–100.
- [DLL] J. Dolbeault, A. Laptev and M. Loss, *Lieb-Thirring inequalities with improved constants*, JEMS, **10** (2008), 1121–1126.
- [Fe] C. L. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), no. 2, 129–206.
- [Fr] R.L. Frank, *Cwikel's theorem and the CLR inequality*, accepted by JST.

- [FL] R.L. Frank and A. Laptev, *Spectral inequalities for Schrödinger operators with surface potentials*, Spectral theory of differential operators, Amer. Math. Soc. Transl. Ser. 2, **225**, Amer. Math. Soc., Providence, RI (2008) 91–102.
- [HLW] D. Hundertmark, A. Laptev and T. Weidl, *New bounds on the Lieb-Thirring constants*, Inv. Math., **140** (2000), 693–704.
- [HLT] D. Hundertmark, E.H. Lieb and L.E. Thomas, *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator*, Adv. Theor. Math. Phys., **2** (1998), 719–731.
- [Ku] P. Kuchment, *Quantum graphs: I. Some basic structures*, Waves in Random Media **14** (2004), S107–128.
- [L] A. Laptev, *Dirichlet and Neumann Eigenvalue Problems on Domains in Euclidean Spaces*, J. Funct. Anal., **151** (1997), 531–545.
- [LW1] A. Laptev and T. Weidl, *Sharp Lieb-Thirring inequalities in high dimensions*, Acta Mathematica, **184** (2000), 87–111.
- [LW2] A. Laptev and T. Weidl, *Recent results on Lieb-Thirring inequalities*, Journées “Équations aux Dérivées Partielles? (La Chapelle sur Erdre, 2000), Exp. No. XX, Univ. Nantes, Nantes, 2000.
- [LY] P. Li and S.-T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys., **88** (1983), 309–318.
- [L] E.H. Lieb, *The Number of Bound States of One-Body Schrödinger Operators and the Weyl Problem*, Proceedings of the Amer. Math. Soc. Symposia in Pure Math., **36** (1980), 241–252 .
- [LT] E.H. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Math. Phys., Essays in Honor of Valentine Bargmann., Princeton, (1976), 269–303.
- [Roz] G.V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, Dokl. AN SSSR, **202** (1972), 1012–1015, Izv. VUZov, Matematika, **1** (1976), 75–86.
- [S] .W. Schmincke, *On Schrödingers factorization method for Sturm-Liouville operators*. Proc. Roy. Soc. Edinburgh Sect. A **80** (1978), 67–84.
- [W1] T. Weidl, *On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$* , Comm. Math. Phys., **178** (1996), 135–146.

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