INEQUALITIES BETWEEN DIRICHLET AND NEUMANN EIGENVALUES ON THE HEISENBERG GROUP

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ABSTRACT. We prove that for any domain in the Heisenberg group the $k+1$ st Neumann eigenvalue of the sub-Laplacian is strictly less than the k th Dirichlet eigenvalue. As a byproduct we obtain similar inequalities for the Euclidean Laplacian with a homogeneous magnetic field.

1. INTRODUCTION AND MAIN RESULT

Universal eigenvalue inequalities are a classical topic in the spectral theory of differential operators. Most relevant to our work here are comparison theorems between the Dirichlet and Neumann eigenvalues $\lambda_j(-\Delta_\Omega^D)$ and $\lambda_j(-\Delta_\Omega^N)$, $j \in \mathbb{N}$, of the Laplacian in a smooth, bounded domain $\Omega \subset \mathbb{R}^d$. Note that $\lambda_j(-\Delta_\Omega^N) \leq \lambda_j(-\Delta_\Omega^D)$ for all $j \in \mathbb{N}$ by the variational characterization of eigenvalues. This trivial bound for $j = 1$ was strengthened by Pólya [Pól] who observed that $\lambda_2(-\Delta_\Omega^N) < \lambda_1(-\Delta_\Omega^D)$ for $d = 2$. Payne [Pay], Aviles [Avi] and Levine and Weinberger [LevWei] obtained further results in this direction under suitable convexity assumptions on Ω. A breakthrough was made by Friedlander [Fri] who proved that

$$\lambda_{j+1}(-\Delta_\Omega^N) \leq \lambda_j(-\Delta_\Omega^D) \qquad \text{for all } j \in \mathbb{N}\,, \tag{1.1}$$

without any curvature assumption on $\partial\Omega$. Later, Filonov [Fil] simplified Friedlander's proof, removed the smoothness assumption on $\partial\Omega$ and showed that (1.1) is strict for $d \geq 2$. While it is still open whether the Payne-Levine-Weinberger bound $\lambda_{j+d}(-\Delta_\Omega^N) \leq \lambda_j(-\Delta_\Omega^D)$ holds for non-convex domains in $\mathbb{R}^d$, the attention has recently shifted to non-Euclidean analogues of (1.1) on Riemannian manifolds. Mazzeo [Maz] has shown for instance that (1.1) holds for domains in hyperbolic space but may fail for domains on the sphere; see also [AshLev] and [HsuWan].

Our goal in this paper to obtain the analogue of (1.1) on the Heisenberg group. In this setting (1.1) was previously known only under rather restrictive and non-generic geometric assumptions on Ω. We have managed to remove these conditions and, as a bonus, obtain similar inequalities for the Euclidean Laplacian with a homogeneous magnetic field.

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The Heisenberg group $\mathbb{H}$ is the prime example of non-commutative harmonic analysis and we refer to [Ste] for background material. We consider $\mathbb{H}$ as $\mathbb{R}^3$ with coordinates (x,y,t) and the (non-commutative) multiplication $(x,y,t)\circ(x',y',t') = (x+x',y+y',t+t'-2(xy'-yx'))$. The vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\,, \qquad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

are left-invariant and the sub-Laplacian on $\mathbb{H}$ is given by

$$-X^2 - Y^2 = -\left(\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}\right)^2 .$$

We are interested in the Dirichlet and Neumann realizations of this sub-Laplacian on domains $\Omega\subset\mathbb{H}$. The space $L_2(\Omega)$ is defined with respect to the restriction to Ω of the Lebesgue measure (which coincides with the Haar measure on $\mathbb{H}$) and hence coincides with its Euclidean counterpart. If Ω is understood, we denote the norm of $u\in L_2(\Omega)$ simply by $\|u\|$. The Sobolev spaces on the Heisenberg group (in this context also known as Folland-Stein spaces) are defined as follows. We denote by $S^1(\Omega)$ the space of all $u\in L_2(\Omega)$ for which the distributional derivatives Xu and Yu belong to $L_2(\Omega)$, equipped with the norm $(\|Xu\|^2+\|Yu\|^2+\|u\|^2)^{1/2}$. The space $\mathring{S}^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $S^1(\Omega)$. The Dirichlet and the Neumann sub-Laplacians L_Ω^D and L_Ω^N on Ω are defined as the self-adjoint operators in $L_2(\Omega)$ corresponding to the quadratic form

$$\|Xu\|^2 + \|Yu\|^2 = \int_\Omega \left(|Xu|^2 + |Yu|^2\right)\,dx\,dy\,dt$$

with form domains $\mathring{S}^1(\Omega)$ and $S^1(\Omega)$, respectively. For any lower semi-bounded operator A with purely discrete spectrum (which is equivalent to its form domain being compactly embedded into the underlying Hilbert space) we denote by $\lambda_j(A)$, $j\in\mathbb{N}$, the j-th eigenvalue of A, counting multiplicities. The variational principle implies immediately the inequality $\lambda_j(L_\Omega^N)\le\lambda_j(L_\Omega^D)$ for all j. Our main result is the analogue of Friedlander's inequality (1.1) on $\mathbb{H}$. We shall prove

Theorem 1.1. *Let $\Omega\subset\mathbb{H}$ be a domain of finite measure such that the embedding $S^1(\Omega)\subset L_2(\Omega)$ is compact. Then $\lambda_{j+1}(L_\Omega^N)<\lambda_j(L_\Omega^D)$ for any $j\in\mathbb{N}$.*

Remark 1.2. The assumption that the embedding $S^1(\Omega)\subset L_2(\Omega)$ is compact can be relaxed. Indeed, our proof shows that if $\Omega\subset\mathbb{H}$ is a non-empty domain of finite measure (which implies that L_Ω^D has discrete spectrum) then the total spectral multiplicity of the operator L_Ω^N in the interval $[0,\lambda_j(L_\Omega^D))$ is at least $j+1$.

Theorem 1.1 holds also on the higher-dimensional Heisenberg groups $\mathbb{H}^{2n+1}$; see Section 3.

We close this introduction by commenting on the similarities and differences between the proofs of (1.1) in the Heisenberg and in the Euclidean case. As emphasized by Mazzeo [Maz], Friedlander's proof of the Euclidean inequality (1.1) relies on the existence, for any $\lambda > 0$, of a function U such that

$$-\Delta U = \lambda U \qquad |\nabla U| \leq \sqrt{\lambda}|U|\,. \tag{1.2}$$

Of course, on Euclidean space such functions are provided by $U(x) = e^{i\sqrt{\lambda}x\cdot\omega}$, $\omega \in \mathbb{S}^{d-1}$. Actually, an inspection of the proofs in [Fri, Fil] shows that the second, pointwise property in (1.2) can be relaxed to the averaged property

$$\int_\Omega |\nabla U|^2\,dx \leq \lambda \int_\Omega |U|^2\,dx\,.$$

Similarly, we will prove Theorem 1.1 by constructing functions U such that

$$-(X^2+Y^2)U = \lambda U\,, \qquad \|XU\|^2_{L_2(\Omega)} + \|YU\|^2_{L_2(\Omega)} \leq \lambda \|U\|^2_{L_2(\Omega)}\,. \tag{1.3}$$

This construction is described in Subsection 2.1 and constitutes the main novelty of this paper. While it is easy to find explicit solutions $U_{z'}$, depending on a parameter $z' \in \mathbb{R}^2$, of the equation in (1.3), it seems rather difficult to prove that for given z' and Ω the inequality in (1.3) is satisfied. Our way around this impasse is to show that the energy inequality holds after *averaging* over $z' \in \mathbb{R}^2$. We believe that this averaging technique might have further applications beyond the present context.

For the sake of clarity we carry out the averaging procedure first for the two-dimensional Landau operator. We emphasize that the connection between this operator and the sub-Laplacian on the Heisenberg group was also essential in the recent proof of sharp Berezin-Li-Yau inequalities on $\mathbb{H}$ [HanLap]; see also [Str]. Eigenvalue inequalities for the Landau operator which we obtain along our way to Theorem 1.1 are presented in the final Section 3.

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2. Proof of Theorem 1.1

2.1. **Eigenfunctions of the two-dimensional Landau operator.** For $z = (x, y) \in \mathbb{R}^2$ let $\mathbf{A}(x,y) := \frac{1}{2}(-y,x)^T$ and $\mathbf{D} = -i\nabla$. For $B > 0$ the spectrum of the self-adjoint operator $(\mathbf{D} - B\mathbf{A})^2$ in $L_2(\mathbb{R}^2)$ consists of the points $B(2k-1)$, $k \in \mathbb{N}$, each being an eigenvalue of infinite multiplicity. Hence there exist infinitely many linearly independent functions U on $\mathbb{R}^2$ satisfying $(\mathbf{D} - B\mathbf{A})^2 U = B(2k-1)U$ and $\int_{\mathbb{R}^2} (\mathbf{D} - B\mathbf{A})U|^2\,dz = B(2k-1)\int_{\mathbb{R}^2} |U|^2\,dz$. It is a non-trivial question, however, whether for a given domain Ω one can find U's such that $\int_\Omega |(\mathbf{D} - B\mathbf{A})U|^2\,dz \leq B(2k-1)\int_\Omega |U|^2\,dz$. That the answer is affirmative is the content of

Proposition 2.1. *Let $B > 0$, $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^2$ a domain of finite measure. There are infinitely many linearly independent functions $U \in C^\infty(\Omega) \cap L_2(\Omega)$ satisfying*

$$(\mathbf{D} - B\mathbf{A})^2 U = B(2k-1)U \qquad \text{in } \Omega\,,$$
$$\int_\Omega |(\mathbf{D} - B\mathbf{A})U|^2\,dz \le B(2k-1)\int_\Omega |U|^2\,dz\,.$$

In order to prove this proposition we use some properties of the spectral projection P_k^B corresponding to the eigenvalue $B(2k-1)$, $k \in \mathbb{N}$, of the operator $(\mathbf{D} - B\mathbf{A})^2$ in $L_2(\mathbb{R}^2)$. This projection is an integral operator with integral kernel

$$P_k^B(z,z') = \frac{B}{2\pi} e^{-iBz\times z'/2 - B|z-z'|^2/4} L_{k-1}(B|z-z'|^2/2)\,. \tag{2.1}$$

We will choose the U's in Theorem 2.1 as $P_k^B(\cdot, z')$ for different values of z'. Indeed, since P_k^B is a projector corresponding to $B(2k-1)$, one has

$$(\mathbf{D}_z - B\mathbf{A}(z))^2 P_k^B(z,z') = B(2k-1)P_k^B(z,z') \tag{2.2}$$

for any z'. In order to find z''s for which the claimed energy bound holds we use the following averaging lemma. It appeared in [Fra] in a different context and we include here a proof for the sake of completeness.

Lemma 2.2. *Let $B > 0$ and $k \in \mathbb{N}$. Then for all $z \in \mathbb{R}^2$*

$$\int_{\mathbb{R}^2} |(\mathbf{D}_z - B\mathbf{A}(z))P_k^B(z,z')|^2\,dz' = B(2k-1)\int_{\mathbb{R}^2} |P_k^B(z,z')|^2\,dz'\,. \tag{2.3}$$

We emphasize that the integration in (2.3) is with respect to the variable z'. The identity is also true (and easier to prove) when the integrals are performed with respect to z with z' fixed. Our proof below does not use the explicit form (2.1), but only that P_k^B is smooth and is constant on the diagonal (which follows by the magnetic translation covariance of the Landau operator).

Proof. We denote $\mathbf{Q}_z := \mathbf{D}_z - B\mathbf{A}(z)$ and abbreviate $P := P_k^B$. Since $P^2 = P$, the left side of (2.3) equals $\mathbf{Q}_z\overline{\mathbf{Q}_{z'}}P(z,z')|_{z=z'}$. Using this and that $P(x,x) = B/2\pi$, the right side equals $B^2(2k-1)/(2\pi)$. Noting that $\mathbf{Q}_z^2 P(z,z') = B(2k-1)P(z,z')$, and hence

$$\mathbf{Q}_z^2 P(z,z')|_{z=z'} = B(2k-1)\frac{B}{2\pi} \qquad \text{and} \qquad \overline{\mathbf{Q}_{z'}}^2 P(z,z')|_{z=z'} = B(2k-1)\frac{B}{2\pi}\,,$$

it suffices to prove that

$$\left(\mathbf{Q}_z^2 + \overline{\mathbf{Q}_z'}^2 - 2\mathbf{Q}_z\overline{\mathbf{Q}_{z'}}\right) P(z,z')|_{z=z'} = 0\,. \tag{2.4}$$

Now we expand $\mathbf{Q}_z$ and $\mathbf{Q}_{z'}$ and write $\mathbf{Q}_z^2 + \overline{\mathbf{Q}_{z'}}^2 - 2\mathbf{Q}_z\overline{\mathbf{Q}_{z'}}$ as a sum of three terms, containing only derivatives of order zero, one and two, respectively. The zeroth order term is easily seen to vanish if $z = z'$. The first order term is given by

$-2B\,(\mathbf{A}(z)-\mathbf{A}(z'))\cdot(\mathbf{D}_z+\mathbf{D}_{z'})$ and hence also vanishes if $z=z'$. Thus (2.4) is equivalent to

$$\left(\mathbf{D}_z^2+\mathbf{D}_{z'}^2+2\mathbf{D}_z\mathbf{D}_{z'}\right)P(z,z')|_{z=z'}=0\,.$$

The latter equality follows by differentiating the identity $P_k(z,z)=\frac{B}{2\pi}$ twice with respect to z. This concludes the proof of (2.3). □

We now turn to the

Proof of Proposition 2.1. Recalling (2.2) we will look for U in the form $P_k^B(\cdot,z')$. According to Lemma 2.2,

$$\int_{\mathbb{R}^2}\int_\Omega |(\mathbf{D}_z-B\mathbf{A}(z))P_k^B(z,z')|^2\,dz\,dz' = B(2k-1)\int_{\mathbb{R}^2}\int_\Omega |P_k^B(z,z')|^2\,dz\,dz'\,.$$

As observed in the proof of that lemma the right hand side equals $B(2k-1)\frac{B}{2\pi}|\Omega|$ and hence both sides are finite. Hence the set K of all $z'\in\mathbb{R}^2$ such that

$$\int_\Omega |(\mathbf{D}_z-B\mathbf{A}(z))P_k^B(z,z')|^2\,dz \le B(2k-1)\int_\Omega |P_k^B(z,z')|^2\,dz \tag{2.5}$$

has positive measure. To complete the proof we have to show that the set $\{\chi_\Omega P_k^B(\cdot,z'):z'\in K\}$ is infinite dimensional.

By Fubini's theorem there is an $a\in\mathbb{R}$ such that $\Gamma:=\{x'\in\mathbb{R}:(x',a)\in K\}$ has positive measure. Let $b\in\mathbb{R}$ such that $I:=\{x\in\mathbb{R}:(x,b)\in\Omega\}$ is non-empty. We claim that the functions $P_k^B((\cdot,b),z')$, $z'\in\Gamma$, are linearly independent on I. Indeed, if

$$\sum_{j=1}^N \alpha_j P_k^B((x,b),w^{(j)})=0 \qquad \text{for all } x\in I$$

and some $\alpha_j\in\mathbb{C}$ and $w^{(j)}=(s^{(j)},a)\in\Gamma$, then by (2.1)

$$\sum_{j=1}^N \tilde\alpha_j e^{B(xs^{(j)}-ixa)/2}L_{k-1}(B((x-s^{(j)})^2+(a-b)^2)/2)=0 \qquad \text{for all } x\in I\,,$$

where $\tilde\alpha_j:=e^{iBbs^{(j)}/2-B(s^{(j)})^2/4}\alpha_j$. Since the left-hand side of this identity is a real-analytic function of x, it holds for all $x\in\mathbb{R}$. Letting $x\to\infty$ one easily concludes that $\tilde\alpha_j=0$ for all j, and hence also $\alpha_j=0$, as claimed. □

Remark 2.3. Proposition 2.1 has a three-dimensional analogue. Indeed, the same proof shows that if $B>0$, $k\in\mathbb{N}$ and $\Omega\subset\mathbb{R}^3$ is a domain of finite measure there exist infinitely many linearly independent functions $U\in C^\infty(\Omega)\cap L_2(\Omega)$, depending only on the variables $(x,y)\in\mathbb{R}^2$, such that

$$(\mathbf{D}_{(x,y)}-B\mathbf{A}(x,y))^2U=B(2k-1)U \qquad \text{in } \Omega\,,$$

$$\int_\Omega |(\mathbf{D}_{(x,y)}-B\mathbf{A}(x,y))U|^2\,dx\,dy\,dt \le B(2k-1)\int_\Omega |U|^2\,dx\,dy\,dt\,.$$

2.2. **Proof of Theorem 1.1.** Given Remark 2.3, Theorem 1.1 follows similarly as in [Fil]. We include the proof not only in order to make this paper self-contained, but also since we have managed to simplify Filonov's proof by avoiding the use of a unique continuation result.

We abbreviate $\lambda_j^D := \lambda_j(L_\Omega^D)$ and similarly for the Neumann eigenvalues. Let $j \in \mathbb{N}$ be fixed and denote by $\varphi_1^D, \ldots, \varphi_j^D$ orthonormal eigenfunctions corresponding to the eigenvalues $\lambda_1^D, \ldots, \lambda_j^D$. Moreover, we choose $k \in \mathbb{N}$ and $\tau > 0$ such that $4\tau(2k-1) = \lambda_j^D$. According to Remark 2.3 there exists a smooth function U on Ω depending only on the variables (x, y) such that

$$
\begin{aligned}
(\mathbf{D}_{(x,y)} - 4\tau\mathbf{A}(x,y))^2 U &= 4\tau(2k-1)U \qquad \text{in } \Omega\,, \\
\int_\Omega |(\mathbf{D}_{(x,y)} - 4\tau\mathbf{A}(x,y))U|^2 \, dx\, dy\, dt &\leq 4\tau(2k-1) \int_\Omega |U|^2 \, dx\, dy\, dt\,.
\end{aligned}
\tag{2.6}
$$

and such that $e^{i\tau t}U$ is linearly independent of $\varphi_1^D, \ldots, \varphi_j^D$ and of the space $\mathcal{N}$ spanned by all Neumann eigenfunctions corresponding to eigenvalues less or equal to λ_{j+1}^N. (We emphasize that if λ_{j+1}^N is degenerate, the dimension of $\mathcal{N}$ might exceed $j+1$, but is finite by the compactness assumption.) With this choice of U the space

$$
\mathcal{M} := \operatorname{span}\{\varphi_1^D, \ldots, \varphi_j^D, e^{i\tau t}U\}
$$

is $j+1$-dimensional and hence by the variational principle

$$
\lambda_{j+1}^N \leq \sup_{0 \not\equiv u \in \mathcal{M}} \frac{\|Xu\|^2 + \|Yu\|^2}{\|u\|^2}\,. \tag{2.7}
$$

In order to estimate the Rayleigh quotient we write an arbitrary $u \in \mathcal{M}$ as

$$
u(x,y,t) := \sum_{i=1}^{j} \alpha_i \varphi_i^D(x,y,t) + \alpha_{j+1} e^{i\tau t} U(x,y)
$$

with constants $\alpha_1, \ldots, \alpha_{j+1} \in \mathbb{C}$. Using the equation of the φ_i^D and their orthogonality we obtain

$$
\begin{aligned}
\|Xu\|^2 + \|Yu\|^2 = &\sum_{i=1}^{j} \lambda_i^D |\alpha_i|^2 + |\alpha_{j+1}|^2 \int_\Omega \left(\left|Xe^{i\tau t}U\right|^2 + \left|Ye^{i\tau t}U\right|^2 \right) dx\, dy\, dt \\
&+ 2\operatorname{Re} \sum_{i=1}^{j} \overline{\alpha_{j+1}} \alpha_i \int_\Omega \left(\overline{Xe^{i\tau t}U} X\varphi_i^D + \overline{Ye^{i\tau t}U} Y\varphi_i^D \right) dx\, dy\, dt\,.
\end{aligned}
$$

Note that $(Xe^{i\tau t}U, Ye^{i\tau t}U)^T = ie^{i\tau t}(\mathbf{D}_{(x,y)} - 4\tau\mathbf{A}(x,y))U$. Integrating by parts, using that φ_i^D satisfies Dirichlet boundary conditions and recalling the equation in (2.6) for U yields

$$
\begin{aligned}
&\int_\Omega \left(\overline{Xe^{i\tau t}U} X\varphi_i^D + \overline{Ye^{i\tau t}U} Y\varphi_i^D \right) dx\, dy\, dt \\
&\quad = \int_\Omega e^{-i\tau t} \overline{(\mathbf{D}_{(x,y)} - 4\tau\mathbf{A}(x,y))^2 U}\ \varphi_i^D \, dx\, dy\, dt = 4\tau(2k-1) \int_\Omega e^{-i\tau t} \overline{U} \varphi_i^D \, dx\, dy\, dt\,.
\end{aligned}
$$

Moreover, by the estimate in (2.6)

$$\int_\Omega \left(\left|Xe^{i\tau t}U\right|^2 + \left|Ye^{i\tau t}U\right|^2 \right) dx\,dy\,dt$$

$$= \int_\Omega |(\mathbf{D}_{(x,y)} - 4\tau\mathbf{A}(x,y))U|^2\,dx\,dy\,dt \le 4\tau(2k-1)\int_\Omega |U(x,y)|^2\,dx\,dy\,dt\,.$$

Hence, estimating $\lambda_i^D \le \lambda_j^D$ and recalling that $4\tau(2k-1) = \lambda_j^D$ we obtain

$$\|Xu\|^2 + \|Yu\|^2 \le \lambda_j^D \Bigg(\sum_{i=1}^j |\alpha_i|^2 + 2\,\mathrm{Re} \sum_{i=1}^j \overline{\alpha_{j+1}}\alpha_i \int_\Omega e^{-i\tau t}\overline{U}\varphi_i^D\,dx\,dy\,dt$$

$$+|\alpha_{j+1}|^2 \int_\Omega |U(x,y)|^2\,dx\,dy\,dt \Bigg)$$

$$= \lambda_j^D \|u\|^2\,.$$

By the variational principle, see (2.7), this implies that $\lambda_{j+1}^N \le \lambda_j^D$. Moreover, the inequality in (2.7) is strict unless $\mathcal{M} \subset \mathcal{N}$. But this is impossible since we have chosen $e^{i\tau t}U$ to be linearly independent of $\mathcal{N}$. This proves Theorem 1.1.

3. Two extensions

3.1. **The Landau operator.** In this subsection we let $d = 2$ or $d = 3$. If $d = 2$ we use coordinates $z = (x, y)$ and define $\mathbf{A}(x,y) := \frac{1}{2}(-y,x)^T$. If $d = 3$ we use coordinates $z = (x,y,t)$ and define $\mathbf{A}(x,y,t) := \frac{1}{2}(-y,x,0)^T$. For a domain $\Omega \subset \mathbb{R}^d$ we put $H^1_{B\mathbf{A}}(\Omega) := \{u \in L_2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega) : (\mathbf{D} - \mathbf{A})u \in L_2(\Omega)\}$ with norm $(\|(\mathbf{D} - B\mathbf{A})u\|^2 + \|u\|^2)^{1/2}$ and denote by $\mathring{H}^1_{B\mathbf{A}}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^1_{B\mathbf{A}}(\Omega)$. The self-adjoint operators $H^D_\Omega(B)$ and $H^N_\Omega(B)$ in $L_2(\Omega)$ are defined via the quadratic forms

$$\|(\mathbf{D} - B\mathbf{A})u\|^2 = \int_\Omega |(\mathbf{D} - B\mathbf{A})u|^2\,dz$$

with form domains $\mathring{H}^1_{B\mathbf{A}}(\Omega)$ and $H^1_{B\mathbf{A}}(\Omega)$, respectively.

Theorem 3.1. *Let $B > 0$ and let $\Omega \subset \mathbb{R}^2$, $d = 2, 3$, be a domain of finite measure such that the embedding $H^1_{B\mathbf{A}}(\Omega) \subset L_2(\Omega)$ is compact.*

1. *If $d = 2$ let $k \in \mathbb{N}$ and assume that $H^D_\Omega(B)$ has j eigenvalues less or equal to $B(2k-1)$. Then $H^N_\Omega(B)$ has $j+1$ eigenvalues less than $B(2k-1)$.*
2. *If $d = 3$ then $\lambda_{j+1}(H^N_\Omega(B)) < \lambda_j(H^D_\Omega(B))$ for all $j \in \mathbb{N}$.*

For $d = 2$ this result is only meaningful for $k \ge 2$, since the spectrum of $H^D_\Omega(B)$ starts above B. Note that by the diamagnetic inequality $|(\mathbf{D} - B\mathbf{A})u| \ge |\nabla|u||$ the compactness of $H^1(\Omega) \subset L_2(\Omega)$ is sufficient for the compactness of $H^1_{B\mathbf{A}}(\Omega) \subset L_2(\Omega)$.

Proof. First assume that $d = 2$. Let $\varphi_1^D, \ldots, \varphi_j^D$ be the Dirichlet eigenfunctions corresponding to the eigenvalues less or equal $B(2k-1)$ and let $\mathcal{N}$ be the subspace generated by the Neumann eigenfunctions corresponding to the eigenvalues less or equal $B(2k-1)$. Let U be a function as in Proposition 2.1 which is linearly independent of $\varphi_1^D,\dots,\varphi_j^D$ and $\mathcal{N}$. Then any function u in the span of $\varphi_1^D,\dots,\varphi_j^D$ and U satisfies by a similar calculation as in the proof of Theorem 1.1

$$\int_\Omega |(\mathbf{D}-B\mathbf{A})u|^2\,dx\,dy \le B(2k-1)\int_\Omega |u|^2\,dx\,dy\,.$$

Hence the $(j+1)$ st Neumann eigenvalue is less or equal to $B(2k-1)$, and equality is excluded as before by linear independence of U and $\mathcal{N}$.

Now let $d=3$. Let $\varphi_1^D,\dots,\varphi_j^D$ be Dirichlet eigenfunctions corresponding to the eigenvalues $\lambda_i^D := \lambda_i(H_\Omega^D(B))$, $i=1,\dots,j$, and let $\mathcal{N}$ be the subspace generated by the Neumann eigenfunctions corresponding to the eigenvalues less or equal $\lambda_j^N := \lambda_i(H_\Omega^N(B))$. Since $\lambda_j^D \ge \lambda_1^D \ge B$ we can choose $k\in\mathbb{N}$ and $\tau\in\mathbb{R}$ such that $B(2k-1)+\tau^2=\lambda_j^D$. Let U be a function as in Remark 2.3 such that $e^{i\tau t}U$ is linearly independent of $\varphi_1^D,\dots,\varphi_j^D$ and $\mathcal{N}$. Using that

$$\begin{aligned}\int_\Omega |(\mathbf{D}_z-B\mathbf{A}(z))e^{i\tau t}U|^2\,dz &= \int_\Omega \left(|(\mathbf{D}_{(x,y)}-B\mathbf{A}(x,y))U|^2+\tau^2|U|^2\right)\,dz\\ &\le \left(B(2k-1)+\tau^2\right)\int_\Omega |e^{i\tau t}U|^2\,dz\end{aligned}$$

one finds that any function u in the span of $\varphi_1^D,\dots,\varphi_j^D$ and $e^{i\tau t}U$ satisfies

$$\int_\Omega |(\mathbf{D}-B\mathbf{A})u|^2\,dz \le \lambda_j^D\int_\Omega |u|^2\,dz$$

and one derives the asserted inequality as before. □

Robin boundary conditions

3.2. **Higher dimensional Heisenberg groups.** $\mathbb{H}^{2n+1}$

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