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Hardy and Sobolev inequalities on antisymmetric functions

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We obtain sharp Hardy inequalities on antisymmetric functions, where antisymmetry is understood for multi-dimensional particles. Partially it is an extension of the paper [Th. Hoffmann-Ostenhof and A. Laptev, Hardy inequalities with homogeneous weights, *J. Funct. Anal.* **268** (2015) 3278–3289], where Hardy's inequalities were considered for the antisymmetric functions in the case of the 1D particles. As a byproduct we obtain

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some Sobolev and Gagliardo–Nirenberg type inequalities that are applied to the study of spectral properties of Schrödinger operators.

Keywords: Hardy inequalities; antisymmetric functions; Sobolev inequalities; Caffarelli–Kohn–Nirenberg inequality.

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1. Introduction

The classical Hardy inequality reads for $u \in \mathcal{H}^1(\mathbb{R}^n)$, $n \geq 3$,

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} dx. \tag{1.1}$$

The literature concerning different versions of Hardy inequalities and their applications is extensive and we are not able to cover it in this paper. We just mention the classical paper [3] and books [2, 5, 6, 9, 22]. Clearly, if $n = 2$, then the Hardy inequality (1.1) is trivial.

For $n \geq 3$ we also have the classical Sobolev inequality

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \geq S(n) \left(\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad u \in \mathcal{H}^1(\mathbb{R}^n), \tag{1.2}$$

where

$$S(n) = \frac{n(n-2)}{4} |\mathbb{S}^n|^{2/n} = \frac{n(n-2)}{4} 2^{2/n} \pi^{1+1/n} \Gamma\left(\frac{n+1}{2}\right)^{-2/n}. \tag{1.3}$$

The inequalities (1.1) and (1.2) are related. In [10], the authors proved that (1.1) implies (1.2) and the Sobolev inequality (1.2) implies only a weak version of the Hardy's inequality (see also [2, 25]). Hardy and Sobolev inequalities are also closely related to spectral properties of the negative eigenvalues of Schrödinger operators.

The aim of this paper is to consider functional and spectral inequalities and their relations to antisymmetric functions.

Let N and d be natural numbers. We consider $x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$, where $x_i = (x_{i1}, \ldots, x_{id}) \in \mathbb{R}^d$ for all $1 \leq i \leq N$. Every function u defined on $\mathbb{R}^{dN}$ we call *antisymmetric* hereafter, if for all $1 \leq i, j \leq N$ and $x_1, \ldots, x_N \in \mathbb{R}^d$

$$u(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = -u(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N).$$

Let us consider the subclass of antisymmetric functions from $\mathcal{H}^1(\mathbb{R}^{dN})$, that we denote by $\mathcal{H}^1_A(\mathbb{R}^{dN})$. Clearly, $\mathcal{H}^1_A(\mathbb{R}^{dN}) \subset \mathcal{H}^1(\mathbb{R}^{dN})$ and therefore it is expected that the constants in the inequalities (1.1) and (1.2) are larger.

Let $\mathcal{V}_d(N)$ be the degree of the Vandermonde determinant defined in (2.1). Among the main results obtained in paper is the following theorem.

Theorem 1. *Let $d, N \in \mathbb{N}$ and let $d \geq 1$, $N \geq 2$. For any $u \in \mathcal{H}_A^1(\mathbb{R}^{dN})$ we have*

$$\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 \, dx \geq H_A(dN) \int_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} \, dx, \tag{1.4}$$

where

$$H_A(dN) = \frac{(dN-2)^2}{4} + \mathcal{V}_d(N)(\mathcal{V}_d(N) + dN - 2). \tag{1.5}$$

At the end of Sec. 3 we shall show that the constant (1.5) in (1.4) is sharp. The constant $(dN-2)^2/4$ in (1.5) is the classical Hardy constant and the constant $\mathcal{V}_d(N)(\mathcal{V}_d(N) + dN - 2)$ is related to the antisymmetry of the class of functions $\mathcal{H}_A^1(\mathbb{R}^{dN})$, with $\mathcal{V}_d(N)$ defined in Sec. 2.

The following theorem gives an improvement of the constant in the Sobolev inequality restricted to antisymmetric functions.

Theorem 2. *For every $u \in \mathcal{H}_A^1(\mathbb{R}^{dN})$, $dN \geq 3$,*

$$\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx \geq S_A(dN) \left(\int_{\mathbb{R}^{dN}} |u(x)|^{\frac{2dN}{dN-2}} dx \right)^{\frac{dN-2}{dN}},$$

where

$$S_A(dN) = (N!)^{\frac{2}{dN}} S(dN) = \pi dN(dN-2) \left(\frac{\Gamma(\frac{dN}{2}) N!}{\Gamma(dN)} \right)^{\frac{2}{dN}}$$

and where $S(dN)$ is the classical Sobolev constant in $\mathbb{R}^{dN}$.

The constant $S_A(dN)$ is sharp and substantially larger than the classical constant $S(dN)$.

Note that the inequality (1.4) in the case $d = 1$ and arbitrary N has been obtained in [15] with the sharp constant $H_A(N) = \frac{(N^2-2)^2}{4}$, $N \geq 2$. Comparing this with the classical constant $(N-2)^2/4 \sim N^2$ as $N \to \infty$, the constant $H_A(N) \sim N^4$. The proof of Theorem 1 in the case $d = 1$ is based on the lowest eigenvalue of the Laplace–Beltrami operator on spherical harmonics generated by antisymmetric harmonic polynomials. The case $d > 1$ is more delicate and is related to the proof obtained in [14] on the absence of the bound states at the threshold in the triplet S-sector for Schrödinger operators defined on a class of antisymmetric functions and where some properties of fermionic wavefunctions were considered. We show in Sec. 3 that for a fixed d we have $H_A(dN) \sim N^{2+2/d}$, as $N \to \infty$.

Some related inequalities were obtained in [13, 21], where it was proved that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx$$

and where $u(x) = -u(-x) \in \mathcal{H}^1(\mathbb{R}^N)$, $N \geq 2$. If $d = 1$ and $N = 2$ then the constant in this inequality coincides with $H_A(2) = 1$.

In [23], the author has obtained Hardy's and Sobolev's inequalities in cones, where the constants in Hardy's inequalities depend on the lowest Dirichlet eigenvalue of the Laplace–Beltrami operator defined on the intersection of $\mathbb{S}^{N-1}$ and the cone in $\mathbb{R}^N$. In our case such eigenvalues can be computed explicitly due to the properties of antisymmetric functions.

The proof of the Sobolev inequality (2) is related to a split of the space, where the antisymmetric function have the same absolute values. After that in each part of the split one can use isoperimetric type inequalities related rearrangements.

In Sec. 2, we study some properties of Vandermonde determinants. In Secs. 3 and 4, we prove Theorem 1 and Theorem 2, respectively. We apply our results to the study of spectral properties of Schrödinger operators in Sec. 5. Finally, in Sec. 6 we present a table with some eigenvalues of the Laplace–Beltrami operator on antisymmetric functions defined on $\mathbb{S}^{dN-1}$.

2. On Vandermonde Type Determinants

2.1. *Some preliminary results*

Let us consider the unitary monomials of d variables lexicographically, i.e. for $t \in \mathbb{R}^d$ denote $\varphi_1^{(d)}(t) = 1$, $\varphi_2^{(d)}(t) = t_1$ and $\varphi_{d+1}^{(d)}(t) = t_d$, $\varphi_{d+2}^{(d)}(t) = t_1^2$ and so on. We now consider the determinant

$$\psi_N^{(d)}(x_1, \ldots, x_N) = \begin{vmatrix} \varphi_1^{(d)}(x_1) & \cdots & \varphi_1^{(d)}(x_N) \\ \vdots & \ddots & \vdots \\ \varphi_N^{(d)}(x_1) & \cdots & \varphi_N^{(d)}(x_N) \end{vmatrix}. \tag{2.1}$$

Let $\mathcal{V}_d(N)$ be the degree of $\psi_N^{(d)}$. The total degree of the above determinant is equal to the sum of degrees in every row.

Proposition 1. *For $N \geq 2$ the polynomial $\psi_N^{(d)}(x_1, \ldots, x_N)$ is an antisymmetric, homogeneous and harmonic function.*

Proof. The first two statements are obvious. In order to proof the harmonicity we use induction. Let d be fixed. The base of induction is obvious, because the degree of $\psi_2^{(d)}(x_1, x_2)$ equals 1. Using induction we find

$$\begin{aligned} \Delta\psi_N^{(d)}(x_1, \ldots, x_N) &= (-1)^N \Delta(\varphi_N^{(d)}(x_1)\psi_{N-1}^{(d)}(x_2, \ldots, x_N) \\ &\quad - \cdots (-1)^N \varphi_N^{(d)}(x_N)\psi_{N-1}^{(d)}(x_1, \ldots, x_{N-1})) \\ &= \begin{vmatrix} \varphi_1^{(d)}(x_1) & \cdots & \varphi_1^{(d)}(x_N) \\ \vdots & \ddots & \vdots \\ \Delta_1\varphi_N^{(d)}(x_1) & \cdots & \Delta_N\varphi_N^{(d)}(x_N) \end{vmatrix}. \end{aligned}$$

It suffices to note that $\Delta_i \varphi_N^{(d)}(x_i)$ is a linear combination of $\varphi_1^{(d)}(x_i), \ldots, \varphi_{N-1}^{(d)}(x_i)$ for all $1 \leq i \leq N$, because it has a smaller degree. The proof is complete. □

Proposition 2. *Let $P(x_1, \ldots, x_N)$ be an antisymmetric homogeneous polynomial. Then* $\deg P \geq \deg \psi_N^{(d)}$.

Proof. Consider P as the sum of monomials

$$P(x_1, \ldots, x_N) = \sum_i a_i Q_i(x_1, \ldots, x_N) = \sum_i a_i \prod_{j=1}^{N} q_{ij}(x_j),$$

where $a_i \in \mathbb{C}\backslash\{0\}$ and $q_{ij}(t) = q_{ij}(t_1, \ldots, t_d) = t_1^{\alpha_{ij1}} \ldots t_d^{\alpha_{ijd}}$ for some $\alpha_{ijk} \in \mathbb{N}_0$. Since P is homogeneous, every Q_i has the same degree as P. Also for all i the monomials q_{ij} are pairwise distinct. Indeed, if $q_{ij_1} = q_{ij_2}$, then

$$Q_i(x_1, \ldots, x_{j_1}, \ldots, x_{j_2}, \ldots, x_n) = Q_i(x_1, \ldots, x_{j_2}, \ldots, x_{j_1}, \ldots, x_n)$$

and

$$P(x_1, \ldots, x_{j_1}, \ldots, x_{j_2}, \ldots, x_n) = -P(x_1, \ldots, x_{j_2}, \ldots, x_{j_1}, \ldots, x_n).$$

Due to equality of these polynomials we obtain the equality of respective coefficients and conclude that $a_i = 0$. Proposition 2 is proved. □

Propositions 1 and 2 demonstrate that functions $\psi_N^{(d)}$ defined above is the minimal antisymmetric harmonic homogeneous polynomial.

2.2. *Exact expression of the determinant degree*

For further considerations we need to prove an auxiliary combinatorial fact.

Lemma 1. *For all $n \in \mathbb{N} = \{n\}_{n=1}^{\infty}$ and $m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ the following equality holds true*

$$\sum_{k=0}^{m} \binom{n+k}{k} = \sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}. \tag{2.2}$$

Proof. Let us prove this proposition by using induction. For $m = 0$ Eq. (2.2) is obvious. Under the assumption that (2.2) is true for $m = m_0$, it is easy to see that

$$\sum_{k=0}^{m_0+1} \binom{n+k}{k} = \binom{n+m_0+1}{n+1} + \binom{n+m_0+1}{n} = \binom{n+m_0+2}{n+1}.$$

This proves the lemma. □

Let $K_p^{(d)}$ be the number of different unitary monomials of degree p of d variables. Note that

$$K_p^{(d)} = \#\{\alpha = (\alpha_1, \ldots, \alpha_d) : \alpha_1 + \cdots + \alpha_d = p,\ \alpha_j \geq 0\}.$$

Proposition 3. *For $p \in \mathbb{N}_0$ and $d \in \mathbb{N}$ we have the following equality*

$$K_p^{(d)} = \binom{p+d-1}{d-1}.$$

Proof. As before we use induction but now with respect to d. It is obvious that for $d = 1$ and any p we have

$$K_p^{(1)} = \#\{\alpha = (\alpha_1) : \alpha_1 = p,\ \alpha_1 \geq 0\} = 1 = \binom{p}{0}.$$

To prove the induction step one can check that

$$K_p^{(d)} = \sum_{\alpha_d=0}^{p} K_{p-\alpha_d}^{(d-1)}.$$

Under the induction assumption we have

$$K_p^{(d)} = \sum_{\alpha_d=0}^{p} \binom{p-\alpha_d+d-2}{d-2} \underset{\beta_d=p-\alpha_d}{=} \sum_{\beta_d=0}^{p} \binom{\beta_d+d-2}{d-2}.$$

According to Lemma 1, we conclude that

$$K_p^{(d)} = \binom{p+d-1}{d-1},$$

and this ends the proof. □

For some special cases N the matrix contains all monomials with degree less or equal m. For a fixed $m \geq 0$ such N we denote by $N_m^{(d)}$. Obviously,

$$N_m^{(d)} = \sum_{p=0}^{m} K_p^{(d)} = \sum_{p=0}^{m} \binom{p+d-1}{d-1} = \binom{m+d}{d} = \frac{(m+1)\dots(m+d)}{d!}.$$

The last equality is true due to Lemma 1. Note that it implies that

$$N_{m+1}^{(d)} = \frac{m+d+1}{m+1} N_m^{(d)} \quad \text{and} \quad N_{m+1}^{(d)} - N_m^{(d)} = \frac{d}{m+1} N_m^{(d)}.$$

We now calculate the degree of $\psi_N^{(d)}$. For the special cases

$$\mathcal{V}_d(N_m^{(d)}) = \sum_{p=0}^{m} pK_p^{(d)} = \sum_{p=1}^{m} p\binom{p+d-1}{d-1} = \sum_{p=1}^{m} d\binom{p+d-1}{d}$$

$$= d\sum_{p=0}^{m-1} \binom{p+d}{d} = d\binom{m+d}{d+1} = \frac{d\,m}{d+1}\binom{m+d}{d} = \frac{d\,m}{d+1} N_m^{(d)}.$$

For the intermediate cases with $N_m^{(d)} < N < N_{m+1}^{(d)}$

$$\mathcal{V}_d(N) = \mathcal{V}_d(N_m^{(d)}) + (N - N_m^{(d)})(m+1)$$

$$= N_m^{(d)}\left(\frac{dm}{d+1} - m - 1\right) + N(m+1)$$

$$= N(m+1) - \frac{m+d+1}{d+1}N_m^{(d)}.$$

Note that for $N = N_m^{(d)}$ and $N = N_{m+1}^{(d)}$ the value $\mathcal{V}_d(N)$ coincides with $\mathcal{V}_d(N_m^{(d)})$ and $\mathcal{V}_d(N_{m+1}^{(d)})$, respectively. In fact,

$$N_{m+1}^{(d)}(m+1) - \frac{m+d+1}{d+1}N_m^{(d)} = N_{m+1}^{(d)}(m+1) - \frac{m+1}{d+1}N_{m+1}^{(d)}$$

$$= \frac{(m+1)d}{d+1}N_{m+1}^{(d)} = \mathcal{V}_d(N_{m+1}^{(d)}).$$

2.3. *Estimates of $\mathcal{V}_d(N)$*

In order to estimate $\mathcal{V}_d(N)$ we need some auxiliary facts. Denote

$$A^{(d)}(m) = \frac{(m+1) + (m+2) + \cdots + (m+d)}{d}$$

and

$$G^{(d)}(m) = \sqrt[d]{(m+1)\ldots(m+d)}$$

for some $m, d \in \mathbb{N}$. These are expressions for arithmetic and geometric means. It is well known that $A^{(d)}(m) \geq G^{(d)}(m)$.

Lemma 2. *For all $m, d \in \mathbb{N}$ we have*

$$G^{(d)}(m) \geq \sqrt{(m+1)(m+d)}.$$

Proof. Let $t = m + \frac{d+1}{2}$. If d is odd, then we have that for k, s.t. $d = 2k+1$

$$(G^{(d)}(m))^d = (t-k)(t-(k-1))\ldots(t+k)$$

$$= t(t^2-1)(t^2-4)\ldots(t^2-k^2) \geq t(t^2-k^2)^k.$$

According to the remark above, $G^{(d)}(m) \leq t$. Therefore

$$(G^{(d)}(m))^{2k} \geq (t^2-k^2)^k \quad \text{and} \quad G^{(d)}(m) \geq \sqrt{(m+1)(m+d)}.$$

If d is even, then we assume that $d = 2k$ and hence

$$(G^{(d)}(m))^d = \left(t - k + \frac{1}{2}\right)\left(t - \left(k - \frac{3}{2}\right)\right)\ldots\left(t + k - \frac{1}{2}\right)$$

$$= \left(t^2 - \frac{1}{4}\right)\left(t^2 - \frac{9}{4}\right)\ldots\left(t^2 - \left(k - \frac{1}{2}\right)^2\right)$$

$$\geq \left(t^2 - \left(k - \frac{1}{2}\right)^2\right)^k.$$

□

Lemma 3. *For all $d \in \mathbb{N}$ there is a constant $C_d \in \mathbb{R}$ such that*

$$0 \le A^{(d)}(m) - G^{(d)}(m) < \frac{C_d}{m}.$$

Proof. Due to the inequality on arithmetic and geometric means $A^{(d)}(m) - G^{(d)}(m) \ge 0$. However, according to the previous lemma,

$$\begin{aligned} A^{(d)}(m) - G^{(d)}(m) &\le m + \frac{d+1}{2} - \sqrt{(m+1)(m+d)} \\ &= \frac{(m + \frac{d+1}{2})^2 - (m+1)(m+d)}{m + \frac{d+1}{2} + \sqrt{(m+1)(m+d)}} \\ &= \frac{(d-1)^2}{4m + 2(d+1) + 4\sqrt{(m+1)(m+d)}}. \end{aligned}$$

The last expression implies that it suffices to let C_d be equal $\frac{(d-1)^2}{8}$. □

Theorem 3. *Let $d \in \mathbb{N}$ and $\mathcal{V}_d(N) = \deg \psi_N^{(d)}$. Then*

$$\mathcal{V}_d(N) = \frac{d}{d+1} \sqrt[d]{d!}\, N^{1+\frac{1}{d}} - \frac{d}{2} N + O(N^{1-\frac{1}{d}}),$$

as $N \to \infty$.

Proof. Let us denote

$$\xi_d(N) = \frac{d}{d+1} \sqrt[d]{d!}\, N^{1+\frac{1}{d}} - \frac{d}{2} N. \tag{2.3}$$

Considering it as a function of the continuous variable N we obviously have $\xi_d'(N) = \sqrt[d]{d! N} - \frac{d}{2}$ and since ξ_d' is increasing we conclude that ξ_d is convex.

First, consider the difference between $\mathcal{V}_d(N)$ and $\xi_d(N)$ at special points.

$$\begin{aligned} \mathcal{V}_d(N_m^{(d)}) - \xi_d(N_m^{(d)}) &= \frac{dm}{d+1} N_m^{(d)} - \frac{d}{d+1} \sqrt[d]{d!} (N_m^{(d)})^{1+\frac{1}{d}} + \frac{d}{2} N_m^{(d)} \\ &= N_m^{(d)} \left(\frac{dm}{d+1} - \frac{d}{d+1} \sqrt[d]{d! N_m^{(d)}} + \frac{d}{2} \right) \\ &= \frac{d}{d+1} N_m^{(d)} \left(m + \frac{d+1}{2} - \sqrt[d]{(m+1)\ldots(m+d)} \right) \\ &= \frac{d}{d+1} N_m^{(d)} (A^{(d)}(m) - G^{(d)}(m)). \end{aligned}$$

Consequently, the difference $\mathcal{V}_d(N_m^{(d)}) - \xi_d(N_m^{(d)}) = o(N_m^{(d)})$ and positive. Due to the convexity of $\xi_d(N)$ and the linearity of $\mathcal{V}_d(N)$ on the segment $[N_m^{(d)}, N_{m+1}^{(d)}]$, the value $\xi_d(N)$ does not exceed $\mathcal{V}_d(N)$ for all N.

Second, for all $N \ge N_m^{(d)}$ we find

$$\xi_d(N) \ge \xi_d(N_m^{(d)}) + (N - N_m^{(d)}) \xi_d'(N_m^{(d)}).$$

Hence, for $N \in [N_m^{(d)}, N_{m+1}^{(d)})$

$$\begin{aligned} 0 &\le \mathcal{V}_d(N) - \xi_d(N) \\ &\le \mathcal{V}_d(N_m^{(d)}) + (N - N_m^{(d)})(m+1) - \xi_d(N_m^{(d)}) - (N - N_m^{(d)})\xi_d'(N_m^{(d)}) \\ &= \frac{d}{d+1} N_m^{(d)}(A^{(d)}(m) - G^{(d)}(m)) + (N - N_m^{(d)})\left(m + 1 - \sqrt[d]{d! N_m^{(d)}} + \frac{d}{2}\right). \end{aligned}$$

Finally, we note that $1 \ge \frac{N_m^{(d)}}{N} \ge \frac{N_m^{(d)}}{N_{m+1}^{(d)}} = \frac{m+1}{m+d+1}$ and consequently,

$$\begin{aligned} 0 &\le \frac{\mathcal{V}_d(N) - \xi_d(N)}{N} \\ &\le \frac{d}{d+1}\frac{N_m^{(d)}}{N}(A^{(d)}(m) - G^{(d)}(m)) + \left(1 - \frac{N_m^{(d)}}{N}\right)\left(A^{(d)}(m) - G^{(d)}(m) + \frac{1}{2}\right) \\ &\le \frac{d}{d+1}(A^{(d)}(m) - G^{(d)}(m)) + \frac{d}{m+d+1}\left(A^{(d)}(m) - G^{(d)}(m) + \frac{1}{2}\right) \\ &\le \frac{d}{d+1}\frac{C_d}{m} + \frac{d}{m+d+1}\left(\frac{C_d}{m} + \frac{1}{2}\right) \le \frac{D_d}{m+d+1}, \end{aligned}$$

where $D_d = \frac{d(2d+3)}{d+1} C_d + \frac{d}{2}$. Since

$$(m+d+1)^d > d! N_{m+1}^{(d)} > d! N,$$

we have

$$\begin{aligned} 0 \le \mathcal{V}_d(N) - \xi_d(N) &\le \frac{N D_d}{\sqrt[d]{d! N}} \\ &= \frac{1}{\sqrt[d]{d!}}\left(\frac{d}{2} + \frac{d(d-1)^2(2d+3)}{8(d+1)}\right) N^{1-\frac{1}{d}}. \end{aligned} \tag{2.4}$$

□

Remark 1. For $d = 1$ the estimate (2.4) is exact $\mathcal{V}_1(N) = \xi_1(N) = \frac{N^2 - N}{2}$. It follows from the equality of $A^{(1)}(m)$ and $G^{(1)}(m)$.

Remark 2. For $d = 2$ and $d = 3$ we have

$$\mathcal{V}_2(N) = \frac{2\sqrt{2}}{3} N^{\frac{3}{2}} - N + O(N^{\frac{1}{2}})$$

and

$$\mathcal{V}_3(N) = \frac{3\sqrt[3]{6}}{4} N^{\frac{4}{3}} - \frac{3}{2} N + O(N^{\frac{2}{3}}).$$

Remark 3. Due to the proof of Theorem 3 we conclude

$$\mathcal{V}_d(N) \ge \frac{d}{d+1}\sqrt[d]{d!}\, N^{1+\frac{1}{d}} - \frac{d}{2} N.$$

3. Hardy Inequality

3.1. *Laplace–Beltrami operator on* $\mathbb{S}^{M-1}$

For $M \geq 2$ the Laplacian can be written in polar coordinates (r, θ), where $r = |x|$ and θ is an angular component of $x \in \mathbb{R}^M$, in the following way

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{M-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_\theta.$$

The operator Δ_θ is the Laplace–Beltrami operator on $\mathbb{S}^{M-1}$. It is well known, that the harmonic homogeneous polynomials are connected with the spherical harmonics, which are the eigenfunctions of the Laplace–Beltrami operator $-\Delta_\theta$. To be more precise, let ψ be the harmonic homogeneous polynomial of the degree P and $\psi_\theta = \frac{\psi}{r^M}$. Then

$$-\Delta_\theta \psi_\theta = P(P + M - 2)\psi_\theta.$$

Proposition 4. *Let $dN \geq 3$. The Laplace–Beltrami operator $-\Delta_\theta$ defined on antisymmetric functions from $L^2(\mathbb{S}^{dN-1})$ satisfies the inequality*

$$-\Delta_\theta \geq \lambda_d(N)$$

in the quadratic form sense, where $\lambda_d(N) = \mathcal{V}_d(N)(\mathcal{V}_d(N) + Nd - 2)$.

Proof. Let $\mathcal{B}$ be the orthonormal system of spherical harmonic functions and let $\mathcal{B}_A \subset \mathcal{B}$ be the orthonormal subset of the set $\mathcal{B}$ that are restrictions of antisymmetric homogeneous harmonic polynomials on $\mathbb{S}^{dN-1}$. For any $u \in \mathcal{H}^1_A(\mathbb{R}^{dN})$ we have

$$u(r, \theta) = \sum_{k:\psi_{\theta,k}\in\mathcal{B}_A} u_k(r)\psi_{\theta,k}(\theta).$$

According to Proposition 2, for all k such that $\psi_{\theta,k} \in \mathcal{B}_A$

$$\deg \psi_k \geq \mathcal{V}_d(N) = \frac{d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} - \frac{d}{2}N + O\left(N^{1-\frac{1}{d}}\right).$$

Hence,

$$\begin{aligned}
&-\int_{\mathbb{S}^{dN-1}} \Delta_\theta u(r,\theta) \cdot u(r,\theta) d\theta \\
&\quad = \sum_{k:\psi_{\theta,k}\in\mathcal{B}_A} |u_k(r)|^2 \int_{\mathbb{S}^{dN-1}} (-\Delta_\theta \psi_{\theta,k}(\theta)) \cdot \psi_{\theta,k}(\theta) d\theta \\
&\quad \geq \sum_{k:\psi_{\theta,k}\in\mathcal{B}_A} |u_k(r)|^2 \mathcal{V}_d(N)(\mathcal{V}_d(N) + dN - 2) \\
&\quad = \lambda_d(N) \int_{\mathbb{S}^{dN-1}} |u(r,\theta)|^2 d\theta.
\end{aligned}$$

□

Remark 4. According to Theorem 3,

$$\begin{aligned}\lambda_d(N) &= \left(\frac{d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} - \frac{d}{2}N + O(N^{1-\frac{1}{d}})\right)\\ &\quad\times\left(\frac{d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} - \frac{d}{2}N + O(N^{1-\frac{1}{d}}) + dN - 2\right)\\ &= \frac{d^2}{(d+1)^2}\sqrt[d]{d!^2}N^{2+\frac{2}{d}} + O(N^2).\end{aligned}$$

Remark 5. According to Remark 3,

$$\begin{aligned}\lambda_d(N) &\geq \left(\frac{d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} - \frac{d}{2}N\right)\\ &\quad\times\left(\frac{d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} - \frac{d}{2}N + dN - 2\right)\\ &= \frac{d^2}{(d+1)^2}\sqrt[d]{d!^2}N^{2+\frac{2}{d}} - \frac{d^2}{4}N^2 - \frac{2d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} + dN.\end{aligned}$$

3.2. *Proof of Theorem 1 (Hardy inequality for antisymmetric functions on $\mathbb{R}^{dN}$)*

Passing to polar coordinates (r, θ), we have

$$\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx = \int_0^\infty \int_{\mathbb{S}^{dN-1}} \left(\left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}|\nabla_\theta u|^2\right) r^{dN-1} d\theta dr.$$

Due to the classical Hardy inequality

$$\int_0^\infty \left|\frac{\partial u}{\partial r}\right|^2 r^{dN-1} dr \geq \frac{(dN-2)^2}{4}\int_0^\infty \frac{|u|^2}{r^2} r^{dN-1} dr. \tag{3.1}$$

Besides, Proposition 4 implies

$$\int_{\mathbb{S}^{dN-1}} |\nabla_\theta u|^2 \, d\theta \geq \lambda_d(N)\int_{\mathbb{S}^{dN-1}} |u|^2 d\theta. \tag{3.2}$$

Finally, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx\\ &\quad\geq \int_{\mathbb{S}^{dN-1}}\int_0^\infty \left(\frac{(dN-2)^2}{4}\frac{|u|^2}{r^2} + \frac{\lambda_d(N)}{r^2}|u|^2\right) r^{dN-1} dr d\theta\\ &\quad= \left(\frac{(dN-2)^2}{4} + \lambda_d(N)\right)\int_{\mathbb{R}^{dN}} \frac{|u|^2}{|x|^2} dx,\end{aligned}$$

where $\lambda_d(N) = \mathcal{V}_d(N)(\mathcal{V}_d(N) + Nd - 2)$.

Remark 6. Using properties of $\mathcal{V}_d(N)$ we find

$$H_A(dN) = \frac{d^2}{(d+1)^2}\sqrt[d]{d!^2}N^{2+\frac{2}{d}} + O(N^2)$$

and

$$\begin{aligned} H_A(dN) &\geq \frac{d^2}{(d+1)^2}\sqrt[d]{d!^2}N^{2+\frac{2}{d}} - \frac{d^2}{4}N^2 - \frac{2d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} + dN + \frac{(dN-2)^2}{4} \\ &= \frac{d^2}{(d+1)^2}\sqrt[d]{d!^2}N^{2+\frac{2}{d}} - \frac{2d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} + 1 \\ &= \left(\frac{d}{d+1}\sqrt[d]{d!}N^{1+\frac{1}{d}} - 1\right)^2. \end{aligned}$$

The constant $H_A(dN)$ in Theorem 1 is sharp.

Proposition 5. *For* $d \geq 1$ *and* $N \geq 2$

$$H_A(dN) = \inf_{\substack{u \in \mathcal{H}^1_A(\mathbb{R}^{dN}) \\ u \neq 0}} \frac{\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx}{\int_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} dx}.$$

Proof. The constant $H_A(dN)$ has two terms $(dN-2)^2/4$ and λ_d. The first one is the classical Hardy constant in (3.1) that is sharp but not achieved. Since the inequality (3.2) is also sharp due to Proposition 4 we complete the proof. □

4. Sobolev Inequality

We now consider the Sobolev inequality on antisymmetric functions. It is well-known that for any $u \in \mathcal{H}^1(\mathbb{R}^n)$, $n \geq 3$

$$\int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \geq S(n)\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}},$$

where

$$S(n) = \pi n(n-2)\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\right)^{\frac{2}{n}}.$$

The same inequality holds for any $u \in \mathcal{H}^1_0(\Omega)$, where $\Omega \subset \mathbb{R}^n$. Before proving our Theorem 2 we need to study some properties of symmetric group acting on $\mathbb{R}^{dN}$.

4.1. *The symmetric group acting on* $\mathbb{R}^{dN}$

Let us consider the action of symmetric group $\mathcal{S}_N$ on the space $\mathbb{R}^{dN}$. For an arbitrary $x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$ and $\sigma \in \mathcal{S}_N$ denote by $\sigma x = (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ the permutation of elements x.

Lemma 4. *Let u be an antisymmetric function on $\mathbb{R}^{dN}$. For an arbitrary $x \in \mathbb{R}^{dN}$ and $\sigma \in \mathcal{S}_N, \sigma \neq id$ there is no continuous path $\Gamma : [0,1] \mapsto \mathbb{R}^{dN}$ such that $\Gamma(0) = x$, $\Gamma(1) = \sigma x$ and $|u(\Gamma(t))| > 0$ for all $t \in [0,1]$.*

Proof. We will show this by contradiction. Let Γ be an appropriate path and $A = \{x \in \mathbb{R}^{dN} : x_i = x_j \text{ for some } 1 \leq i,j \leq N\}$. Since $u(x) = 0$ for all $x \in A$, $\Gamma([0,1]) \cap A = \emptyset$. Then let us consider the action $\mathcal{S}_N$ on $\mathbb{R}^{dN} \backslash A$. Because of lack of fixed points on $R = \mathbb{R}^{dN} \backslash A$, this acting generates the covering $p : R \mapsto R/\mathcal{S}_N$. According to the path lifting property, there are no any paths $\tilde{\Gamma}$ different from Γ such that $p(\Gamma) = p(\tilde{\Gamma})$.

Now we consider sets $E_\tau^k = \{x \in \mathbb{R}^{dN} : x_{\tau(1)k} \leq \cdots \leq x_{\tau(N)k}\}$, where $1 \leq k \leq d$ and $\tau \in \mathcal{S}_N$. Without loss of generality we can assume that $x \in E_{id}^k$ for all k. For every k we can construct the projection mapping Σ^k on the fundamental domain E_{id}^k which maps $x \in E_\tau^k$ to $\tau^{-1}x$. Then if for some k the point σx does not belong to E_{id}^k, then the path $\Sigma^k(\Gamma)$ differs from Γ and $p(\Sigma^k(\Gamma)) = p(\Gamma)$. Consequently, x and σx belong to E_{id}^k for all k at the same time.

It remains to note that the inequalities

$$x_{1k} \leq \cdots \leq x_{Nk},$$
$$x_{\sigma(1)k} \leq \cdots \leq x_{\sigma(N)k}$$

imply the equality of respective parts $x_{ik} = x_{\sigma(i)k}$ for all $1 \leq i \leq N$ and $1 \leq k \leq d$. Since $\sigma \neq id$, there is $1 \leq i \leq N$ such that $\sigma(i) \neq i$ and it contradicts the assumption that $x \notin A$. □

Analogically, let us denote $\sigma E = \{\sigma x : x \in E\}$ for arbitrary $E \subset \mathbb{R}^{dN}$ and $\sigma \in \mathcal{S}_N$. Let $C_{0,A}^\infty(\mathbb{R}^{dN})$ be the class of C_0^∞ antisymmetric functions on $\mathbb{R}^{dN}$.

Theorem 4. *Let $u \in C_{0,A}^\infty(\mathbb{R}^{dN})$. Then there is a set $E \subset \mathbb{R}^{dN}$ such that sets $\{\sigma E\}_{\sigma \in \mathcal{S}_N}$ are disjoint, $u = 0$ for all $x \in \delta E$ and $\mu(\operatorname{supp} u \backslash \bigcup_{\sigma \in \mathcal{S}_N} \sigma E) = 0$.*

Proof. Let us divide $\operatorname{supp} u$ into the union of connected components $\{U_\alpha\}$. Because of their openness, U_α are path-connected and according to Lemma 4, we can consider the group action $\mathcal{S}_N$ on $\{U_\alpha\}$. At the end, it remains to choose representatives from each equivalence class. Their union will be the desired set E. □

4.2. *Proof of Theorem 2 (Sobolev Inequality)*

It is enough to show it for an arbitrary u belonging to the subclass $C_{0,A}^\infty(\mathbb{R}^{dN}) \subset C_0^\infty(\mathbb{R}^{dN})$ — functions satisfying antisymmetry conditions. The case $u \in \mathcal{H}_A^1(\mathbb{R}^{dN})$ follows by the completeness $C_{0,A}^\infty(\mathbb{R}^{dN})$ in $\mathcal{H}_A^1(\mathbb{R}^{dN})$.

Let E be the set from Theorem 4. The restriction of u to the set E satisfies zero boundary conditions at the boundary ∂E. Thus, we have

$$\int_E |\nabla u(x)|^2 dx \geq S(dN) \left(\int_E |u(x)|^{\frac{2dN}{dN-2}} dx \right)^{\frac{dN-2}{dN}}.$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx &= N! \int_E |\nabla u(x)|^2 dx \\ &\geq N!\ S(dN) \left(\int_E |u(x)|^{\frac{2dN}{dN-2}} dx \right)^{\frac{dN-2}{dN}} \\ &= (N!)^{\frac{2}{dN}} S(dN) \left(\int_{\mathbb{R}^{dN}} |u(x)|^{\frac{2dN}{dN-2}} dx \right)^{\frac{dN-2}{dN}},\end{aligned}$$

which proves Theorem 2.

Remark 7. The constant $S_A(dN) = (N!)^{\frac{2}{dN}} S(dN)$ is sharp and substantially larger than the classical one. It is enough to consider the minimizing sequence for the classical Sobolev inequality on E and extend it on $\mathbb{R}^{dN}$ by antisymmetry.

Remark 8. Due to Stirling's approximation we find

$$S_A(dN) \sim \frac{\pi e^{1-\frac{2}{d}}}{2} dN^{1+\frac{2}{d}} \quad \text{as } N \to \infty.$$

5. Applications to Spectral Inequalities

Having two improved classical inequalities (Hardy and Sobolev) we now apply them to spectral properties of Schrödinger operators.

5.1. *Caffarelli–Kohn–Nirenberg type inequality*

Proposition 6. *Let* $p = \frac{2dN}{dN-2\nu}$ *and* $\gamma = 2dN \frac{\nu-1}{dN-2\nu}, 0 \leq \nu \leq 1, dN \geq 3$. *Then for any function* $u \in \mathcal{H}^1_A(\mathbb{R}^{dN})$ *we have*

$$\left(\int_{\mathbb{R}^{dN}} |x|^\gamma |u(x)|^p dx \right)^{\frac{2}{p}} \leq \widetilde{\mathcal{K}}(dN, \nu) \left(\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx \right)^\nu \left(\int_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} dx \right)^{1-\nu},$$

where $\widetilde{\mathcal{K}}_d(N,\nu) = S_A^{-\nu}(dN)$ *and where* $S_A^\nu(dN)$ *is defined in Theorem* 2.

Proof.

$$\begin{aligned}&\left(\int_{\mathbb{R}^{dN}} |x|^\gamma |u(x)|^p dx \right)^{\frac{2}{p}} \\ &\quad = \left(\int_{\mathbb{R}^{dN}} \left(\frac{|u(x)|}{|x|} \right)^{(1-\nu)p} |u(x)|^{p\nu} dx \right)^{\frac{2}{p}}\end{aligned}$$

$$\leq \left(\int_{\mathbb{R}^{dN}} |u(x)|^{\frac{2dN}{dN-2}} dx\right)^{\frac{\nu(dN-2)}{dN}} \left(\int_{\mathbb{R}^{dN}} \left(\frac{|u(x)|}{|x|}\right)^2 dx\right)^{1-\nu}$$

$$\leq S_A^{-\nu}(dN) \left(\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx\right)^{\nu} \left(\int_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} dx\right)^{1-\nu}. \qquad \square$$

Proposition 7. *Let* $p = \frac{2dN}{dN-2\nu}$ *and* $\gamma = 2dN\frac{\nu-1}{dN-2\nu}, 0 \leq \nu \leq 1, dN \geq 3$. *Then for any antisymmetric function* $u \in \mathcal{H}_A^1(\mathbb{R}^{dN})$ *we have*

$$\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx \geq \mathcal{K}_d(N,\nu) \left(\int_{\mathbb{R}^{dN}} |x|^{\gamma} |u(x)|^p dx\right)^{\frac{2}{p}},$$

where $\mathcal{K}_d(N,\nu) = S_A^{\nu}(dN)\, H_A^{1-\nu}(dN)$ *and where* $H_A(dN)$ *is given in* (1.5).

Proof.

$$\left(\int_{\mathbb{R}^{dN}} |x|^{\gamma} |u(x)|^p dx\right)^{\frac{2}{p}} \leq S_A^{-\nu}(dN) \left(\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx\right)^{\nu} \left(\int_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} dx\right)^{1-\nu}$$

$$\leq \frac{1}{S_A^{\nu}(dN) H_A^{1-\nu}(dN)} \int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx. \qquad \square$$

Remark 9. Note that as $N \to \infty$ we have

$$\mathcal{K}_d(N,\nu) \sim \left(\frac{\pi e^{1-\frac{2}{d}}}{2}\right)^{\nu} \frac{d^{2-\nu}}{(d+1)^{2-2\nu}} \sqrt[d]{d!^{2-2\nu}}\, N^{2+\frac{2}{d}-\nu}.$$

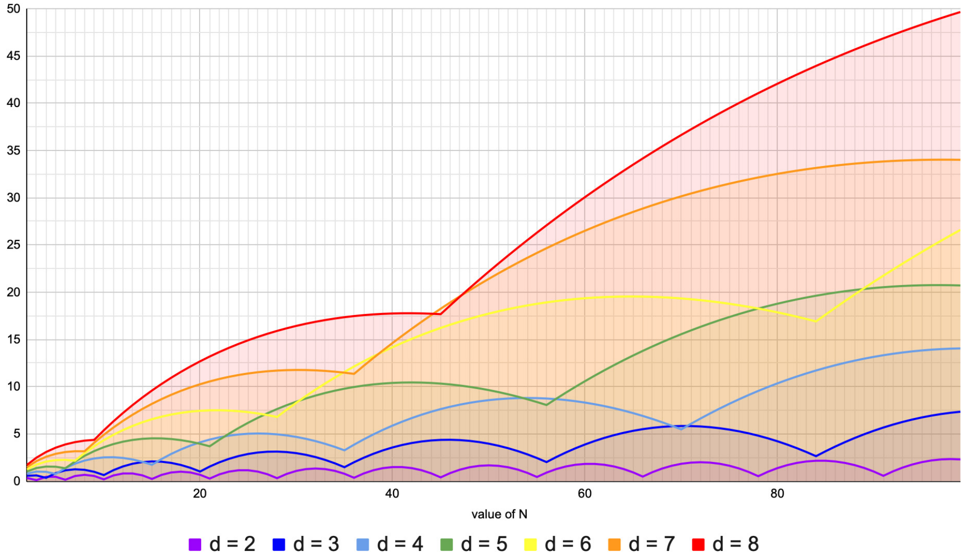

Fig. 1. The values of $\mathcal{V}_d(N) - \xi_d(N)$, $2 \leq d \leq 8$.

5.2. *Spectral properties of Schrödinger operators*

Let us consider a Schrödinger operator defined on antisymmetric functions in $L^2_A(\mathbb{R}^{dN})$

$$\mathscr{H} = -\Delta - V,$$

where $V \geq 0$.

Theorem 5. *Let $dN \geq 3$ and $0 \leq \nu \leq 1$. Assume that*

$$\left(\int_{\mathbb{R}^{dN}} V^{\frac{dN}{2\nu}} |x|^{\frac{1-\nu}{\nu} dN} dx\right)^{\frac{2\nu}{dN}} \leq \mathcal{K}_d(N,\nu).$$

Then the operator $\mathscr{H}$ is positive and has no negative eigenvalues.

Proof. For the quadratic form of the operator $\mathscr{H}$ we find

$$\begin{aligned}
&\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^{dN}} V(x)|u(x)|^2 dx \\
&\quad = \int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^{dN}} V(x)|x|^{2(1-\nu)}|u(x)|^2|x|^{2(\nu-1)} dx \\
&\quad \geq \int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx - \left(\int_{\mathbb{R}^{dN}} V(x)^{\frac{dN}{2\nu}} |x|^{\frac{1-\nu}{\nu} dN} dx\right)^{\frac{2\nu}{dN}} \\
&\qquad \times \left(\int_{\mathbb{R}^{dN}} |u(x)|^{\frac{2dN}{dN-2\nu}} |x|^{2dN\frac{\nu-1}{dN-2\nu}} dx\right)^{\frac{dN-2\nu}{dN}} dx \\
&\quad \geq \mathcal{K}_d(N,\nu) \left(\int_{\mathbb{R}^{dN}} |x|^\gamma |u(x)|^p dx\right)^{\frac{2}{p}} \\
&\qquad - \mathcal{K}_d(N,\nu) \left(\int_{\mathbb{R}^{dN}} |u(x)|^p |x|^\gamma dx\right)^{\frac{2}{p}} dx = 0,
\end{aligned}$$

where $p = \frac{2dN}{dN-2\nu}$ and $\gamma = 2dN\frac{\nu-1}{dN-2\nu}$. Consequently, $\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^{dN}} V(x)|u(x)|^2 dx \geq 0$ and this completes the proof. □

6. Some Numerical Values

Despite of complexity of exact values of the degree $\mathcal{V}_d(N)$ of the Vandermonde determinant (2.1) we can give some values of minimal eigenvalues of the

Laplace–Beltrami operator obtained with numerics.

	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$
$d=1$	1	12	48	130	285	546	952	1548
$d=2$	3	12	40	84	144	253	392	561
$d=3$	5	18	39	90	161	252	363	494
$d=4$	7	24	51	88	168	272	400	552
$d=5$	9	30	63	108	165	280	423	594
$d=6$	11	36	75	128	195	276	432	620

Minimal eigenvalues of the Laplace–Beltrami operator on $\mathbb{S}^{dN-1}$ on antisymmetric functions.

Also it allows us to compare $\mathcal{V}_d(N)$ and its estimate $\xi_d(N)$ defined in (2.3) that we obtained in (2.4). Results are performed by following graph, where the reader can see the difference $\mathcal{V}_d(N) - \xi_d(N)$ with N growing from 2 to 100 and for $2 \leq d \leq 8$:

Due to (2.4) the difference between $\mathcal{V}_d(N)$ and $\xi_d(N)$ equals $O(N^{1-\frac{1}{d}})$. It justifies the growth of the difference with increasing of d. Also, we can see the special values $N_m^{(d)}$. They correspond to cusps on graphs.

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