HARDY INEQUALITY FOR ANTISYMMETRIC FUNCTIONS

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ABSTRACT. We consider Hardy inequalities on antisymmetric functions. Such inequalities have substantially better constants. We show that they depend on the lowest degree of an antisymmetric harmonic polynomial. This allows us to obtain some Caffarelli-Kohn-Nirenberg-type inequalities that are useful for studying spectral properties of Schrödinger operators.

To Mikhail Zakharovich Solomyak, a colleague and the teacher, with respect and admiration

1. INTRODUCTION

The classical Hardy inequality reads for $u \in H^1(\mathbb{R}^N)$, $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2|}{|x|^2} dx. \tag{1.1}$$

The literature concerning different versions of Hardys inequalities and their applications is extensive and we are not able to cover it in this paper. We just mention the classical paper [3] and books [1], [5], [6], [19].
If $N = 2$ then Hardy type inequalities and their applications to spectral theory of Schrödinger operators were studied in a series of papers of M.Solomyak. Here we just mentions two papers [22], [23].

In this note we consider the inequality (1.1) on a class of antisymmetric functions from $H^1(\mathbb{R}^N)$ that we denote by $H^1_A(\mathbb{R}^N)$. It is assumed that such functions satisfying the following antisymmetry conditions:

$$u(\dots, x_i, \dots, x_j, \dots) = -u(\dots, x_j, \dots, x_i, \dots), \tag{1.2}$$

where $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$.

Clearly $H^1_A(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ and therefore it is expected that the constant in (1.1) is larger. Our goal is to show that for $u \in H_A(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq C_A(N) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx, \qquad N \geq 2, \tag{1.3}$$

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with the explicit constant $C_A(N) = \frac{(N^2-2)^2}{4}$, $N \geq 2$.
Note that some related sharp inequalities were obtained in [11], [18], where it was proved that

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx,$$

where $u(x) = -u(-x) \in H^1(\mathbb{R}^N)$, $N \geq 2$. If $N = 2$ then the constant in this inequality coincides with $C_A(N)$ and equals 1.
The inequalities obtained in Proposition 4 and Corollary 3 for $N = 2$ are limiting cases for Caffarelli-Kohn-Nirenberg [4] inequalities that do not hold without the antisymmetry conditions.

Further applications of Hardy inequalities on antisymmetric functions are used for proving spectral properties of Schrödinger operators with decaying potentials. In particular, we find some new conditions under which Schrödinger operators defined on antisymmetric functions do not have negative eigenvalues. The respective estimates are given in terms of weighted Egorov-Kondrat'ev type norms of potentials, see [9].

The fact that the spectral point zero is not a resonance state of a magnetic Schrödinger operator with Aharonov-Bohm magnetic field in the 2D case, was considered in [2]. For such operators there is a non-trivial Hardy inequality [17] and in [2] the authors have obtained some spectral inequalities in terms of norms of potentials functions in $L^1(\mathbb{R}_+, L^\infty(\mathbb{S}), rdr)$, see also [16]. In the present paper we use 2D-Hardy inequality for antisymmetric functions that allows us to show absence of negative eigenvalues for Schrödinger operators in terms of the Egorov-Kondrat'ev type classes $L^{\varkappa_1}(\mathbb{R}^N, |x|^{\varkappa_2} dx)$ of the potential with some $\varkappa_1$, $\varkappa_2$, see Theorem 2. Such classes were also considered in proving Lieb-Thirring inequalities for Schrödinger operators with subtracted Hardy terms in [7].

Note that in [20] (Proposition 4.1) the author obtained a Hardy inequality related the inequality obtained in this paper, see Theorem 1, Section 3. The constant in the inequality obtained in [20] depends on the lowest Dirichlet eigenvalue of the Laplace-Beltrami operator defined on the intersection of $\mathbb{S}^{N-1}$ and a cone in $\mathbb{R}^N$. In our case such an eigenvalue can be computed explicitly due to the special structure of antisymmetric functions.

Finally we would like to mention the paper [10], where the authors have proved the absence of the bound states at the threshold in the triplet S-sector for Schrödinger operators defined on a class of antisymmetric function.
Some properties of fermionic wave functions were considered in [12].

We begin with some simple statements regarding properties of harmonic polynomials in Section 2. In Section 3 we prove Hardy inequalities on the class of antisymmetric functions. In Section 4 we consider a version of the Caffarelli-Kohn-Nirenberg inequality, where antisymmetry allows us to obtain better constants in Corollary 2. Finally we apply our results to spectral properties of Schrödinger operators in Section 5.

2. Spherical harmonics

2.1. **Laplace-Beltrami operator on $\mathbb{S}^{N-1}$.** The Laplacian in polar coordinates (r, θ), where $\theta = x/r$, $r = |x|$, $x \in \mathbb{R}^N$, equals

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_\theta,$$

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$. A harmonic homogeneous polynomial of degree M is denoted by $P_M(x)$ and the associated spherical harmonic equals $Y_M = \frac{P_M}{r^M}$. It is well known that the spherical harmonics Y_M are eigenfunctions of the Laplace-Beltrami operator $-\Delta_\theta$,

$$-\Delta_\theta Y_M = M(M+N-2)Y_M = \lambda_{M,N} Y_M.$$

The value

$$h(M,N) = \binom{N+M-1}{N-1} - \binom{N+M-3}{N-1}$$

is the multiplicity of the eigenvalue $\lambda_{M,N}$.

2.2. **Properties of analytic antisymmetric functions.** We have the following result (see [13])

Proposition 1. *Let ψ be an analytic antisymmetric function in $\mathbb{R}^N$ satisfying the property* (1.2). *Then there is a symmetric analytic function φ such that $\psi = \varphi\, \mathcal{V}_N$, where $\mathcal{V}_N$ is the Vandermonde determinant*

$$\mathcal{V}_N = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_N \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_N^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \dots & x_N^{N-1} \end{vmatrix}. \tag{2.1}$$

Proof. Let $x = (x_1, x_2, \dots x_N) \in \mathbb{R}^N$. Since ψ is antisymmetric we have $\psi(x) = 0$ for $x_k = x_j$, $k \neq j$. Hence due to analyticity, ψ has factors $x_k - x_j$ for all $k \neq j$. Factorising them out we therefore conclude that the function

$$\varphi(x) = \frac{\psi(x)}{\prod_{k<j}(x_k - x_j)} = \psi(x)\,(\mathcal{V}_N)^{-1}$$

is symmetric and analytic. □

Definition 1. *Denote by $M(N)$, $N > 1$, the smallest integer so that there is an antisymmetric harmonic homogeneous polynomial $P_{M(N)} \not\equiv 0$ with degree $M(N)$.*

The Vandermonde determinant $\mathcal{V}_N$ defined in (2.1) is such a harmonic polynomial whose degree equals $\frac{N(N-1)}{2}$ and thus

$$M(N) = \frac{N(N-1)}{2}. \tag{2.2}$$

Corollary 1. *The Laplace-Beltrami operator Δ_θ defined on antisymmetric functions on $L^2(\mathbb{S}^{N-1})$ satisfies the inequality*

$$-\Delta_\theta \geq M(N) = \frac{N(N-1)}{2}$$

in the quadratic form sense.

3. HARDY'S INEQUALITIES

One of our main results is the following:

Theorem 1. *Let $u \in H^1_A(\mathbb{R}^N)$, $N \geq 2$. Then*

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq C_A(N) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx, \tag{3.1}$$

where

$$C_A(N) = \frac{(N^2-2)^2}{4}. \tag{3.2}$$

Proof. Consider polar coordinates $x = (r, \theta)$, $r \in (0, \infty)$, and $\theta \in \mathbb{S}^{N-1}$. Then

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \int_0^\infty \int_{\mathbb{S}^{N-1}} \left(\left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \right) r^{N-1}\, d\theta dr. \tag{3.3}$$

Let $\mathcal{Y}$ be the orthonormal system of spherical harmonic functions and let $\mathcal{Y}_A \subset \mathcal{Y}$ be the orthonormal subset of the set $\mathcal{Y}$ that are restrictions of totally antisymmetric homogeneous harmonic polynomials. For any $u \in H^1_A(\mathbb{R}^N)$ we have

$$u(r, \theta) = \sum_{k:Y_k \in \mathcal{Y}_A} u_k(r) Y_k(\theta).$$

Then using that $M(N) = \min\{k : Y_k \in \mathcal{Y}_A\}$ and that

$$\lambda_{M,N} = M(N)(M(N) + N - 2) = \frac{N(N-1)(N^2+N-4)}{4}$$

we find

$$\int_{\mathbb{S}^{N-1}} |\nabla_\theta u(r,\theta)|^2\, d\theta = \sum_{k=M(N)}^{\infty} \lambda_k |u_k(r)|^2$$
$$\geq \lambda_{M,N} \sum_{k=M(N)}^{\infty} |u_k(r)|^2 = \lambda_{M,N} \int_{\mathbb{S}^{N-1}} |u(r,\theta)|^2\, d\theta. \quad (3.4)$$

For the radial part we use the classical Hardy inequality on the half-line

$$\int_0^\infty \left|\frac{\partial u}{\partial r}\right|^2 r^{N-1}\, dr \geq \frac{(N-2)^2}{4} \int_0^\infty \frac{|u|^2}{r^2}\, r^{N-1}\, dr. \quad (3.5)$$

Substituting the inequalities (3.4) and (3.5) into (3.3) we finally arrive at

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx \geq \int_0^\infty \int_{\mathbb{S}^{N-1}} \left(\left|\frac{\partial u}{\partial r}\right|^2 + \lambda_{M,N} \frac{|u(r,\theta)|^2}{r^2}\right) r^{N-1}\, d\theta dr$$
$$\geq C_A(N) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\, dx,$$

where $C_A(N) = \lambda_{M,N} + \frac{(N-2)^2}{4} = \frac{(N^2-2)^2}{4}$. □

Remark 1. *It is easy to check that if $N = 2$ then the lowest eigenvalue on the Laplace-Beltrami operator on the circle equals one and therefore for functions $u(x_1, x_2) = -u(x_2, x_1)$ we have*

$$\int_{\mathbb{R}^2} |\nabla u|^2\, dx \geq \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\, dx.$$

Remark 2. *Note that the constant $C_A(N) \sim N^4/4$ compared with the classical Hardy's constant that grows as $N^2/4$, as $N \to \infty$.*

Proposition 2. *The inequality* (3.1) *is sharp.*

Proof. Indeed, let us consider

$$u_0(x) = \varphi(r) Y_{M(N)}(\theta).$$

Then substituting this function into the quadratic form (3.3) we obtain

$$\int_{\mathbb{R}^N} |\nabla u_0|^2\, dx = \int_0^\infty \int_{\mathbb{S}^{N-1}} \left(\left|\frac{\partial \varphi}{\partial r}\right|^2 |Y_{M(N)}|^2 + \frac{1}{r^2} |\varphi|^2 |\nabla_\theta Y_{M(N)}|^2\right) r^{N-1}\, d\theta dr.$$

It is well known that the inequality

$$\int_0^\infty \left|\frac{\partial \varphi}{\partial r}\right|^2 r^{N-1}\, dr \geq \left(\frac{N-2}{2}\right)^2 \int_0^\infty \frac{|\varphi|^2}{r^2}\, r^{N-1}\, dr \quad (3.6)$$

is sharp. Besides,

$$\int_{\mathbb{S}^{N-1}} \frac{1}{r^2}|\nabla_\theta Y_{M(N)}|^2\, d\theta = \lambda_{M,N} \int_{\mathbb{S}^{N-1}} \frac{1}{r^2}|Y_{M(N)}|^2\, d\theta. \tag{3.7}$$

Combining (3.6) and (3.7) we complete the proof. □

4. Caffarelli-Kohn-Nirenberg type inequalities

Let $N \geq 3$. Then the classical Sobolev inequality states

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}} \leq S_N \int_{\mathbb{R}^N} |\nabla u|^2\, dx, \tag{4.1}$$

where the sharp constant S_N was found in [24], see [14],

$$S_N = \frac{N(N-2)}{4}\, |\mathbb{S}^N|^{2/N} = \frac{N(N-2)}{4}\, 2^{2/N}\pi^{1+1/N}\Gamma\left(\frac{N+1}{2}\right)^{-2/N}. \tag{4.2}$$

In this section we discuss some special cases of Caffarelli-Kohn-Nirenberg inequalities considering $N = 2$ and $N \geq 3$ separately.

Proposition 3. *Let $p = \frac{2N}{N-2\vartheta}$, $\gamma = 2N\,\frac{\vartheta-1}{N-2\vartheta}$, $0 \leq \vartheta \leq 1$, $N \geq 3$. Then for any antisymmetric function $u \in H^1(\mathbb{R}^N)$ we have*

$$\left(\int_{\mathbb{R}^N} |x|^\gamma |u|^p dx\right)^{2/p} \leq C_{N,\vartheta} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^\vartheta \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\, dx\right)^{(1-\vartheta)}, \tag{4.3}$$

where

$$C_{N,\vartheta} \leq S_N^\vartheta. \tag{4.4}$$

Remark 3. *Note that if $\vartheta = 1$, then $\gamma = 0$ and $p = 2^* = 2N/(N-2)$. In this case* (4.3) *is just the Sobolev inequality.*
If $\vartheta = 0$, then $p = 2$, $\gamma = -2$ and in (4.3) *the left hand side coincides with the right hand side with $C_{N,0} = 1$.*

In the case $N = 2$ the inequality (4.3) is not valid for $u \in H^1(\mathbb{R}^2)$ and we prove it for anti-symmetric functions.

Proposition 4. *Let $N = 2$. There exists a positive constant $C_{2,\vartheta} > 0$ such that for any $0 \leq \vartheta < 1$ and $u \in H^1_A(\mathbb{R}^2)$ we have*

$$\left(\int_{\mathbb{R}^2} \frac{|u|^{\frac{2}{1-\vartheta}}}{|x|^2} dx\right)^{1-\vartheta} \leq C_{2,\vartheta} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^\vartheta \left(\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\, dx\right)^{(1-\vartheta)}. \tag{4.5}$$

Proof of Proposition 3. Applying Hölder's and Sobolev's inequalities we find

$$\left(\int_{\mathbb{R}^N} |x|^\gamma |u|^p\,dx\right)^{2/p} = \left(\int_{\mathbb{R}^N} |u|^{p\vartheta}\left(\frac{|u|}{|x|}\right)^{p(1-\vartheta)} dx\right)^{2/p}$$
$$\leq \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx\right)^{\frac{\vartheta(N-2)}{N}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx\right)^{1-\vartheta}$$
$$\leq S_N^\vartheta \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\vartheta} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx\right)^{1-\vartheta}.$$

Note that
$$\frac{p\vartheta(N-2)}{2N} + \frac{p(1-\vartheta)}{2} = p\,\frac{N-2\vartheta}{2N} = 1.$$
The proof is complete.

Proof of Proposition 4. Let $B_\rho = \{0 \leq |x| \leq \rho\}$, $\rho > 0$, and let $u \in H^1_A(\mathbb{R}^2)$ be antisymmetric. Then
$$\bar{u} = \int_{B_\rho} u\,dx = 0.$$
Then using the Gagliardo-Nirenberg interpolation inequality [8], [21], see also [15],[19], with $p = \frac{2}{1-\vartheta}$ we have

$$\int_{B_\rho} |u|^p dx \leq C\left(\int_{B_\rho} |\nabla u|^2 dx\right)^{\vartheta p/2} \left(\int_{B_\rho} |u|^2\,dx\right)^{(1-\vartheta)p/2}$$
$$\leq C\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{\vartheta p/2} \int_{B_\rho} |u|^2\,dx, \quad (4.6)$$

where $0 \leq \vartheta < 1$. By scaling, the inequality (4.6) is independent of the radius ρ of the disc B_ρ and thus $C = C(\vartheta)$.
We now multiply both sides of (4.6) by ρ^{-3} and integrate with respect to ρ over $(0, \infty)$. Then using simple identities
$$\int_0^\infty \rho^{-3} \int_{|x|\leq\rho} |u|^p\,dxd\rho = \frac{1}{2}\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^2}dx,$$
$$\int_0^\infty \rho^{-3} \int_{|x|\leq\rho} |u|^2\,dxd\rho = \frac{1}{2}\int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}dx,$$
we finally arrive at
$$\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^2}dx \leq C\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{\vartheta p/2} \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}dx.$$

This proves (4.5) with $C_{2,\vartheta} = C^{\frac{2}{p}} = C^{1-\vartheta}$.

Open questions.
1. The Sobolev inequality is achieved on spherically symmetric functions. Is it possible to improve the constant S_N (4.2) in the Sobolev inequality considering it on functions from $H_A^1(\mathbb{R}^N)$?
2. It would be interesting to find a direct proof of (4.5) with a sharp constant.

Combining Proposition 3 and Theorem 1 we obtain

Corollary 2. *Let $N \geq 3$, $p = \frac{2N}{N-2\vartheta}$, $0 \leq \vartheta \leq 1$ and $\gamma = 2N\,\frac{\vartheta-1}{N-2\vartheta}$. Then*

$$\frac{C_A(N)^{1-\vartheta}}{C_{N,\vartheta}} \left(\int_{\mathbb{R}^N} |x|^\gamma |u|^p dx \right)^{2/p} \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad u \in H_A^1(\mathbb{R}^N). \tag{4.7}$$

where $C_A(N)$ is defined in (3.2).

Since $C_A(2) = 1$, applying Proposition 4 we have

Corollary 3. *Let $p = \frac{2}{1-\vartheta}$ and $0 \leq \vartheta < 1$. Then*

$$\left(\int_{\mathbb{R}^2} \frac{|u|^{\frac{2}{1-\vartheta}}}{|x|^2} dx \right)^{1-\vartheta} \leq C_{2,\vartheta} \int_{\mathbb{R}^2} |\nabla u|^2 dx, \quad u \in H_A^1(\mathbb{R}^2). \tag{4.8}$$

5. Applications to spectral properties of Schrödinger operators

Let us consider a Schrödinger operator in $L^2(\mathbb{R}^N)$

$$H = -\Delta - V, \qquad V \geq 0,$$

and its quadratic form

$$(Hu, u) = \int_{\mathbb{R}^N} (|\nabla u|^2 - V|u|^2)\, dx. \tag{5.1}$$

Theorem 2. *Let p and ϑ satisfy the conditions from Corollary 2 if $N \geq 3$ and Corollay 3 if $N = 2$. Assume that*

$$\frac{C_{N,\vartheta}}{C_A(N)^{1-\vartheta}} \left(\int_{\mathbb{R}^N} V^{\frac{N}{2\vartheta}}\, |x|^{\frac{1-\vartheta}{2\vartheta}N}\, dx \right)^{\frac{2\vartheta}{N}} \leq 1.$$

Then the operator H is positive,

$$H = -\Delta - V \geq 0. \tag{5.2}$$

In particular, if $N = 2$ then (5.2) *is true assuming*

$$C_{2,\vartheta} \left(\int_{\mathbb{R}^2} V^{\frac{1}{\vartheta}}\, |x|^{\frac{1-\vartheta}{\vartheta}}\, dx \right)^{\vartheta} \leq 1.$$

Proof. Using Hölder's inequality we have

$$\int_{\mathbb{R}^N} V|u|^2\,dx = \int_{\mathbb{R}^N} V|x|^{\alpha}|u|^2|x|^{-\alpha}\,dx$$
$$\leq \left(\int_{\mathbb{R}^N} V^q|x|^{\alpha q}\,dx\right)^{1/q}\left(\int_{\mathbb{R}^N}|x|^{-\alpha p}|u|^p\,dx\right)^{2/p},$$

where $1/q + 2/p = 1$ and thus

$$\frac{1}{q} = 1 - 2\frac{N-2\vartheta}{2N} = \frac{2\vartheta}{N}.$$

Choosing

$$\alpha = -\gamma/p = -2N\frac{\vartheta-1}{N-2\vartheta}\cdot\frac{N-2\vartheta}{2N} = 1-\vartheta$$

we have

$$\alpha q = (1-\vartheta)\frac{N}{2\vartheta}.$$

Using Corollary 2 we find

$$\int_{\mathbb{R}^N}(|\nabla u|^2 - V|u|^2)\,dx$$
$$\geq \left(1 - \frac{C_{N,\vartheta}}{C_A(N)^{1-\vartheta}}\left(\int_{\mathbb{R}^N} V^{\frac{N}{2\vartheta}}\,|x|^{\frac{N(1-\vartheta)}{2\vartheta}}\,dx\right)^{\frac{2\vartheta}{N}}\right)$$
$$\times\int_{\mathbb{R}^N}|\nabla u|^2 dx \geq 0.$$

□

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