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Hardy inequalities with homogeneous weights

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ABSTRACT

In this paper we obtain some sharp Hardy inequalities with weight functions that may admit singularities on the unit sphere. In order to prove the main results of the paper we use some recent sharp inequalities for the lowest eigenvalue of Schrödinger operators on the unit sphere obtained in the paper [3].

1. Introduction

The classical Hardy inequality for the Laplacian in $\mathbb{R}^d$

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx, \quad u \in C_0^\infty(\mathbb{R}^d), \quad d \geq 3, \tag{1.1}$$

is well known and has many elementary proofs. This inequality is not achieved but the constant $(d-2)^2/4$ is sharp. It is often associated with the Heisenberg uncertainty

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principle and plays important role in spectral theory of Schrödinger operators. In particular, this inequality is equivalent to the quadratic form inequality

$$-\Delta - \frac{(d-2)^2}{4}\frac{1}{|x|^2} \geq 0,$$

which states that if $d \geq 3$, then one can subtract a positive operator from the Laplacian so that the difference remains non-negative. The literature devoted to different types of Hardy's inequalities is vast and it is not our aim to cover it in this short paper, but note that the description of other "Hardy weights" is an interesting problem. Here we are dealing with the case, where instead of the spherical symmetrical weight $1/|x|^2$ in the integral on the right hand side of (1.1) we consider a more general class of homogeneous functions of degree -2 which may have singularities along rays starting at the origin.

Namely, in this paper we prove the inequality

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2}\, |u(x)|^2\, dx, \quad u \in C_0^\infty(\mathbb{R}^d), \quad d \geq 3, \tag{1.2}$$

with some $\tau > 0$ for a class of measurable functions Φ defined on $\mathbb{S}^{d-1}$. The theorems proved in this paper are based on the recent inequalities obtained in the paper [3], where the authors have found sharp bounds for the first eigenvalue of a Schrödinger operator on $\mathbb{S}^{d-1}$ using deep results from [1].

In order to formulate our results let us introduce the measure $d\vartheta$ induced by Lebesgue's measure on $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. We denote by $\|\cdot\|_{L^p(\mathbb{S}^{d-1})}$ the quantity

$$\|\Phi\|_{L^p(\mathbb{S}^{d-1})} = \left(\int_{\mathbb{S}^{d-1}} |\Phi(\vartheta)|^p\, d\vartheta\right)^{1/p}.$$

Our first result is:

Theorem 1.1. *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where*

$$p \geq \frac{(d-2)^2}{2(d-1)} + 1. \tag{1.3}$$

Then

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2}\, |u(x)|^2\, dx, \qquad u \in C_0^\infty(\mathbb{R}^d), \tag{1.4}$$

where

$$\tau = \frac{(d-2)^2}{4}\, |\mathbb{S}^{d-1}|^{1/p}\, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}. \tag{1.5}$$

Remark 1.2. For the class of functions Φ satisfying the conditions of the theorem, inequality (1.4) is sharp. Indeed, if $\Phi \equiv 1$, then (1.4) takes the classical sharp form

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx.$$

Remark 1.3. If for example $d = 3$, then the lowest possible value of p that is allowed in Theorem 1.1 equals $5/4$, see (1.3).

Note that the condition on the value of p in (1.3) could be weakened. In our next theorem we consider the values of p smaller than $\frac{(d-2)^2}{2(d-1)} + 1$.

Theorem 1.4. *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where*

$$p \in (1,\, 5/4)\,, \text{ if } d = 3, \quad \text{and} \quad p \in \left[\frac{d-1}{2},\, \frac{(d-2)^2}{2(d-1)} + 1\right)\,, \text{ if } d \geq 4.$$

Then

$$\int_{\mathbb{R}^d} |\nabla u|^2\, dx \geq (1-\nu_0)\, \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\, dx + \tau \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2}\, |u|^2\, dx, \tag{1.6}$$

where

$$\nu_0 = \frac{2(d-1)(p-1)}{(d-2)^2} < 1,$$

and

$$\tau = \nu_0\, \frac{(d-2)^2}{4}\, |\mathbb{S}^{d-1}|^{1/p}\, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}. \tag{1.7}$$

Remark 1.5. The inequality (1.6) is sharp and achieved for the functions $\Phi \equiv$ *const*. Moreover, if $p = \frac{(d-2)^2}{2(d-1)} + 1$, then $\nu_0 = 1$ in (1.6) and this inequality coincides with (1.4).

Remark 1.6. In Theorem 4.1 (see Section 4) we consider the values of p

$$\frac{d-1}{2} < p < \frac{(d-2)^2}{2(d-1)} + 1 \tag{1.8}$$

and obtain an inequality similar to (1.6) with different ranges of τ and ν's. It is interesting that in this case the optimal class of functions Φ does not coincide with constants. It is more convenient for us to formulate and prove the respective result after the proof of Theorems 1.1 and 1.4.

Finally in the last section we obtain a Hardy inequality for fractional powers of the Laplacian. Namely, let us define the quadratic form

$$\int_{\mathbb{R}^d} |\nabla^{\varkappa} u(x)|^2\, dx = (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^{2\varkappa} |\hat{u}(\xi)|^2\, d\xi,$$

where $\hat{u}$ is the Fourier transform of u.

Theorem 1.7. *Let $0 < \varkappa < d/2$ for $d = 1, 2$, and $0 < \varkappa \le 1$ for $d \ge 3$. Assume that $\Phi = \Phi(x/|x|) \ge 0$ is a measurable function defined on $\mathbb{S}^{d-1}$, such that $\Phi \in L^{d/2\varkappa}(\mathbb{S}^{d-1})$. Then*

$$\int_{\mathbb{R}^d} |\nabla^{\varkappa}(x)|^2 \ge \tau \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2} |u(x)|^2\, dx, \tag{1.9}$$

where

$$\tau = 2^{2\varkappa} \frac{\Gamma^2\left((d/2+\varkappa)/2\right)}{\Gamma^2\left((d/2-\varkappa)/2\right)} \left|\mathbb{S}^{d-1}\right|^{2\varkappa/d} \|\Phi\|^{-1}_{L^{d/2\varkappa}(\mathbb{S}^{d-1})}. \tag{1.10}$$

In order to prove this theorem we use fractional Hardy inequalities proved in [5] and [9] (note that $2^{2\varkappa}\,\Gamma^2\left((d/2+\varkappa)/2\right)\,\Gamma^{-2}\left((d/2-\varkappa)/2\right)|_{\varkappa=1} = (d-2)^2/4$).

Remark 1.8. Note, that in the case $\varkappa = 1$ Theorem 1.1 is stronger than Theorem 1.7 since it allows us to have a larger class of functions Φ because of the strict embedding

$$L^{d/2}(\mathbb{S}^{d-1}) \subset L^{\frac{(d-2)^2}{2(d-1)}+1}(\mathbb{S}^{d-1}).$$

Remark 1.9. The constant τ in (1.10) is sharp as it is sharp for $\Phi = \text{const}$.

In the recent paper of B. Devyver, M. Fraas and Y. Pinchover [2] the authors considered a rather general second order operator with variable coefficients and found an optimal weight for the respective Hardy inequality. In particular, such a weight for the Laplacian coincides with $1/|x|^2$.

Our result is different as we deal with the "flat" Laplacian and find a class of weight functions that may have singularities not only at the origin.

2. Auxiliary statements

In order to prove Theorem 1.1 we use a result obtained in [3] which provides a sharp estimate for the first negative eigenvalue λ_1 of the Schrödinger operator in $L^2(\mathbb{S}^{d-1})$,

$$-\Delta_\vartheta - \Phi, \qquad \Phi \ge 0,$$

where $-\Delta_\vartheta$ is the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$. Note that we need it only for the case $d \geq 3$.

Theorem 2.1. *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where $p \in \big((d-1)/2, +\infty\big)$. Then there exists an increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$*

$$\alpha(\mu) = \mu \quad \textit{for any} \quad \mu \in \left[0, \frac{d-1}{2}(p-1)\right], \tag{2.1}$$

and convex if $\mu \in \left(\frac{d-1}{2}(p-1), +\infty\right)$, such that

$$|\lambda_1(-\Delta_\vartheta - \Phi)| \leq \alpha\left(\frac{1}{|\mathbb{S}^{d-1}|^{1/p}}\|\Phi\|_{L^p(\mathbb{S}^{d-1})}\right). \tag{2.2}$$

The estimate (2.2) is optimal in the sense that there exists a non-negative function Φ, such that

$$|\lambda_1(-\Delta_\vartheta - \Phi)| = \alpha\left(\frac{1}{|\mathbb{S}^{d-1}|^{1/p}}\|\Phi\|_{L^p(\mathbb{S}^{d-1})}\right),$$

for any $\mu \in \left(\frac{d-1}{2}(p-1), +\infty\right)$. If $\mu \leq \frac{d-1}{2}(p-1)$, then equality in (2.2) is achieved for constants.

For large values of μ we have

$$\alpha(\mu)^{p-\frac{d-1}{2}} = L^1_{p-\frac{d-1}{2},d-1}\,\mu^p\,(1 + o(1)), \tag{2.3}$$

where $L^1_{\gamma,d-1}$ are the Lieb–Thirring constants appearing in [7] in the inequality for the lowest eigenvalue of a Schrödinger operator in $L^2(\mathbb{R}^{d-1})$.

Moreover, if $p = (d-1)/2$, $d \geq 4$, then (2.2) is satisfied with $\alpha(\mu) = \mu$ for $\mu \in [0, (d-1)(d-3)/2]$.

Note that here the function $\alpha(\mu)$ is invertible and its inverse $\mu(\alpha)$ equals (see [3])

$$\mu(\alpha) = |\mathbb{S}^{d-1}|^{\frac{2}{q}-1} \inf_{u \in H^1(S^{d-1})} \frac{\|\nabla u\|^2_{L^2(\mathbb{S}^{d-1})} + \alpha\,\|u\|^2_{L^2(\mathbb{S}^{d-1})}}{\|u\|^2_{L^q(\mathbb{S}^{d-1})}}, \tag{2.4}$$

where $q \in \left(2, \frac{2(d-1)}{d-3}\right)$ (with $(2,\infty)$ for $d = 3$) and where the values of p and $q/2$ are Hölder conjugates, $q = 2p/(p-1)$. The optimal value in (2.4) is achieved by the unique solution u of the non-linear equation

$$-\Delta u + \alpha\, u - \mu(\alpha)\, u^{q-1} = 0,$$

that for each chosen α also defines the value of $\mu(\alpha)$.

Obviously if $v \equiv c$, $c \in \mathbb{R}$, and $\Phi \geq 0$ is non-trivial, then the quadratic form

$$\int_{\mathbb{S}^{d-1}} \left(|\nabla_\vartheta v|^2 - \Phi |v|^2\right) d\vartheta = -c^2 \int_{\mathbb{S}^{d-1}} \Phi \, d\vartheta < 0.$$

Therefore due to the variational principle the eigenvalue $\lambda_1(-\Delta_\vartheta - \Phi)$ is negative for any nonnegative, non-trivial Φ and consequently the inequality (2.2) is a lower estimate

$$0 \geq \lambda_1(-\Delta_\vartheta - \Phi) \geq -\alpha \left(\frac{1}{|\mathbb{S}^{d-1}|^{1/p}} \|\Phi\|_{L^p(\mathbb{S}^{d-1})}\right) \quad \forall \, \Phi \in L^p(\mathbb{S}^{d-1}). \tag{2.5}$$

If Φ changes sign, the above inequality still holds if Φ is replaced by the positive part Φ_+ of Φ, provided the lowest eigenvalue is negative. We can then write

$$|\lambda_1(-\Delta_\vartheta - \Phi)| \leq \alpha \left(\frac{1}{|\mathbb{S}^{d-1}|^{1/p}} \|\Phi_+\|_{L^p(\mathbb{S}^{d-1})}\right).$$

The expressions for the constants $L^1_{p-\frac{d-1}{2},d}$ in (2.3) are not explicit for $d \geq 3$, but can be given in terms of an optimal constant in some Gagliardo–Nirenberg–Sobolev inequality (see [7] and [3]) in the following way:

Let $q = 2p/(p-1) > 2$ and denote by $\mathsf{K}_{\mathrm{GN}}(q, d-1)$ the optimal constant in the Gagliardo–Nirenberg–Sobolev inequality, given by

$$K_{\mathrm{GN}}(q, d-1) := \inf_{u \in H^1(\mathbb{R}^{d-1}) \setminus \{0\}} \frac{\|\nabla u\|^{2\,\rho}_{L^2(\mathbb{R}^{d-1})} \, \|u\|^{2\,(1-\rho)}_{L^2(\mathbb{R}^{d-1})}}{\|u\|^2_{L^q(\mathbb{R}^{d-1})}},$$

where $\rho = \rho(q, d) = (d-1)\,\frac{q-2}{2\,q}$.

Then

$$L^1_{p-\frac{d-1}{2},d-1} = \left[\rho^{-\rho}\,(1-\rho)^{-\,(1-\rho)}\,\mathsf{K}_{\mathrm{GN}}(q, d-1)\right]^{-p}.$$

Lemma 2.2. *Let $\tau > 0$ and $d \geq 3$. Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \left(\tau\,\Phi(x/|x|) + \lambda_1\left(-\Delta_\vartheta - \tau\,\Phi(x/|x|)\right) + \frac{(d-2)^2}{4}\right) dx. \tag{2.6}$$

Proof. Let $x = (r, \vartheta) \in \mathbb{R}^d$ be polar coordinates in $\mathbb{R}^d$. Then we find

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(|\partial_r u|^2 + \frac{1}{r^2}\,|\nabla_\vartheta u|^2\right) r^{d-1} \, d\vartheta dr. \tag{2.7}$$

Note that according to the classical Hardy inequality for radial functions $f \in C_0^\infty(0, \infty)$ we have

$$\int_0^\infty |f'(r)|^2\, r^{d-1}\, dr \geq \frac{(d-2)^2}{4} \int_0^\infty \frac{|f|^2}{r^2}\, r^{d-1}\, dr.$$

Applying the latter inequality to $u(r, \vartheta)$ for a fixed ϑ and then integrating over $\mathbb{S}^{d-1}$ we obtain

$$\int_{\mathbb{S}^{d-1}} \int_0^\infty |\partial_r u|^2\, r^{d-1}\, dr d\vartheta \geq \frac{(d-2)^2}{4} \int_{\mathbb{S}^{d-1}} \int_0^\infty \frac{|u|^2}{r^2}\, r^{d-1}\, dr d\vartheta. \tag{2.8}$$

Let $\tau > 0$. It follows from Theorem 2.1 that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1}{r^2}\, |\nabla_\vartheta u|^2\, r^{d-1}\, d\vartheta dr &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1}{r^2}\, |\nabla_\vartheta u|^2\, r^{d-1}\, d\vartheta dr \\ &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1}{r^2}\, \tau\, \Phi\, |u|^2\; r^{d-1}\, d\vartheta dr \\ &\quad + \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1}{r^2} \left(|\nabla_\vartheta u|^2 - \tau\, \Phi\, |u|^2\right)\; r^{d-1}\, d\vartheta dr \\ &\geq \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{1}{r^2}\, \left(\tau\, \Phi + \lambda_1(-\Delta_\vartheta - \tau\, \Phi)\right)\, |u|^2\, r^{d-1}\, d\vartheta dr. \end{aligned} \tag{2.9}$$

Putting together (2.7), (2.8) and (2.9) we obtain the statement of the lemma. □

Corollary 2.3. *Let $\tau > 0$ and $d \geq 3$ and let $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where*

$$p \in \big(\max\{1, (d-1)/2\}, +\infty\big).$$

Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \left(\tau\, \Phi(x/|x|) - \alpha(\mu) + \frac{(d-2)^2}{4}\right) dx, \tag{2.10}$$

where

$$\mu = \tau\; |\mathbb{S}^{d-1}|^{-1/p}\; \|\Phi\|_{L^p(\mathbb{S}^{d-1})}.$$

Proof. Indeed, in order to prove (2.10) it is enough to apply the inequality (2.5) estimating the value of $\lambda_1\, (-\Delta_\vartheta - \tau\, \Phi(x/|x|))$ in (2.6). □

3. Proofs of the main results

Proof of Theorem 1.1. The condition

$$p \ge \frac{(d-2)^2}{2(d-1)} + 1 \tag{3.1}$$

implies both

$$p \in \left(\frac{d-1}{2}, \infty\right) \quad \text{and} \quad \frac{(d-2)^2}{4} \le \frac{d-1}{2}(p-1).$$

Due to Theorem 2.1 the convex function $\alpha(\mu) = \mu$ for

$$\mu \in \left[0, \frac{(d-1)(p-1)}{2}\right].$$

Thus if in (2.10) we choose τ according to the equation

$$\alpha(\mu) = \mu = |\mathbb{S}^{d-1}|^{-1/p}\, \tau\, \|\Phi\|_{L^p(\mathbb{S}^{d-1})} = \frac{(d-2)^2}{4},$$

namely

$$\tau = \frac{(d-2)^2}{4}\, |\mathbb{S}^{d-1}|^{1/p}\, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})},$$

then we obtain the statement of Theorem 1.1. □

Proof of Theorem 1.4. When proving Theorem 1.1 we fully compensated the positive term on the right hand side of (2.8). This gave us a restriction on the possible values of p, see (3.1). Assume now that

$$p \in (1, 5/4)\,, \ \text{if } d = 3, \quad \text{and} \quad p \in \left[\frac{d-1}{2}, \frac{(d-2)^2}{2(d-1)} + 1\right), \ \text{if } d \ge 4, \tag{3.2}$$

and choose ν_0 such that

$$\nu_0\, \frac{(d-2)^2}{4} = \frac{(d-1)(p-1)}{2}, \tag{3.3}$$

which gives us the value

$$\nu_0 = \frac{2(d-1)(p-1)}{(d-2)^2} < 1.$$

Then using (2.10) we find

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \left(\tau\, \Phi(x/|x|) - \alpha(\mu) + \frac{(d-2)^2}{4} \right) dx$$
$$= \int_{\mathbb{R}^d} \left(\tau\, \Phi(x/|x|) + (1-\nu_0)\, \frac{(d-2)^2}{4} \right) \frac{|u|^2}{|x|^2}\, dx$$
$$+ \int_{\mathbb{R}^d} \left(\nu_0\, \frac{(d-2)^2}{4} - \alpha(\mu) \right) \frac{|u|^2}{|x|^2}\, dx. \tag{3.4}$$

Due to the choice of p and ν_0 given in (3.2) and (3.3) respectively, we have

$$\alpha(\mu) = \mu = |\mathbb{S}^{d-1}|^{-1/p}\, \tau\, \|\Phi\|_{L^p(\mathbb{S}^{d-1})}.$$

It remains to choose τ according to

$$\tau\, |\mathbb{S}^{d-1}|^{-1/p}\, \|\Phi\|_{L^p(\mathbb{S}^{d-1})} = \nu_0\, \frac{(d-2)^2}{4},$$

namely,

$$\tau = \nu_0\, \frac{(d-2)^2}{4}\, |\mathbb{S}^{d-1}|^{1/p}\, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}.$$

This completes the proof of Theorem 1.4. □

4. Hardy inequalities with $\nu_0 < \nu \leq 1$

As it was mentioned in Remark 1.6, for the values

$$\frac{d-1}{2} < p < \frac{(d-2)^2}{2(d-1)} + 1,$$

we can now consider $\nu : \nu_0 < \nu \leq 1$. Then since

$$\frac{(d-2)^2}{4} > \frac{(d-1)(p-1)}{2}$$

the equation

$$\alpha \left(\tau\, |\mathbb{S}^{d-1}|^{-1/p}\, \|\Phi\|_{L^p(\mathbb{S}^{d-1})} \right) = \nu\, \frac{(d-2)^2}{4}$$

is more complicated, because in this case $\alpha(\mu)$ is non-linear. However, since it is increasing and convex, its inverse $\mu(\alpha)$ is well defined and thus we find

$$\tau = |\mathbb{S}^{d-1}|^{1/p}\, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}\, \mu \left(\nu\, \frac{(d-2)^2}{4} \right).$$

Hence the inequality (2.10) immediately implies:

Theorem 4.1. *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where*

$$\frac{d-1}{2} < p < \frac{(d-2)^2}{2(d-1)} + 1.$$

Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq (1-\nu) \, \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx + \tau \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2} \, |u|^2 \, dx, \tag{4.1}$$

where

$$\nu_0 = \frac{2(d-1)(p-1)}{(d-2)^2} < \nu \leq 1$$

and

$$\tau = |\mathbb{S}^{d-1}|^{1/p} \, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})} \, \mu\left(\nu \, \frac{(d-2)^2}{4}\right).$$

Remark 4.2. Note that since $\mu(\alpha)$ is an increasing function, the value of τ in (4.1) is larger than the respective value of τ in (1.6). In particular, $\nu = 1$ allows us to consider a class of weight functions Φ with full compensation of the term $(d-2)^2/4$. If follows from [3] that the optimal functions Φ are not constants.

Remark 4.3. Eq. (2.3) immediately implies

$$\mu(\alpha) = \left(L^1_{p-\frac{d-1}{2},d-1}\right)^{-1/p} \alpha^{1-\frac{d-1}{2p}} \, (1 + o(1)) \quad \text{as} \quad \alpha \to \infty$$

(see also [3, Proposition 10]).

5. Proof of Theorem 1.7

Let $A \subset \mathbb{R}^d$ and denote by $A^* = \{x : |x| < r\}$ with $(|\mathbb{S}^{d-1}|/d)|x|^d = |A|$ that is the symmetric rearrangement of A. By χ_A and χ_{A^*} we denote characteristic functions of A and A^* respectively. Then for any Borel measurable function $f : \mathbb{R}^d \to \mathbb{C}$ vanishing at infinity we denote by f^* its decreasing rearrangement

$$f^*(x) = \int_0^\infty \chi_{\{|f(x)|>t\}^*} \, dt.$$

By using the Hardy–Littlewood rearrangement inequality we find

$$\int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^{2\varkappa}} |u|^2 \, dx \le \int_{\mathbb{R}^d} \left(\frac{\Phi(x/|x|)}{|x|^{2\varkappa}} \right)^* (u^*)^2 \, dx.$$

Clearly

$$\left| \{ x : |\Phi(x/|x|)| > t \, |x|^{2\varkappa} \} \right| = \frac{1}{d} \, t^{-d/2\varkappa} \int_{\mathbb{S}^{d-1}} \Phi^{d/2\varkappa}(\theta) \, d\theta$$

and thus

$$\begin{aligned} \left(\frac{\Phi(x/|x|)}{|x|^{2\varkappa}} \right)^* &= \int_0^\infty \chi_{\{|\Phi(x/|x|)| > t \, |x|^{2\varkappa}\}^*} \, dt \\ &= \int_0^\infty \chi_{\{|\mathbb{S}^{d-1}| \, |x|^d < \int_{\mathbb{S}^{d-1}} \Phi^{d/2\varkappa}(\theta) \, d\theta \, t^{-d/2\varkappa}\}} \, dt \\ &= \frac{1}{|\mathbb{S}^{d-1}|^{2\varkappa/d}} \, \frac{\left(\int_{\mathbb{S}^{d-1}} \Phi^{d/2\varkappa}(\theta) \, d\theta \right)^{2\varkappa/d}}{|x|^{2\varkappa}}. \end{aligned}$$

We now use the Hardy inequality obtained in the papers [5,9] (see also [4] for L^p-versions of these inequalities) stating that if $\varkappa < d/2$, then

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\varkappa}} \, dx \le C_\varkappa \int_{\mathbb{R}^d} |\nabla^\varkappa u|^2 \, dx,$$

where

$$C_\varkappa = 2^{-2\varkappa} \frac{\Gamma^2 \left((d/2 - \varkappa)/2 \right)}{\Gamma^2 \left((d/2 + \varkappa)/2 \right)}.$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^{2\varkappa}} |u|^2 \, dx &\le \int_{\mathbb{R}^d} \left(\frac{\Phi(x/|x|)}{|x|^{2\varkappa}} \right)^* (u^*)^2 \, dx \\ &= \frac{\|\Phi\|_{L^{d/2\varkappa}(\mathbb{S}^{d-1})}}{|\mathbb{S}^{d-1}|^{2\varkappa/d}} \int_{\mathbb{R}^d} \frac{(u^*)^2}{|x|^{2\varkappa}} \, dx \\ &\le C_\varkappa \frac{\|\Phi\|_{L^{d/2\varkappa}(\mathbb{S}^{d-1})}}{|\mathbb{S}^{d-1}|^{2\varkappa/d}} \int_{\mathbb{R}^d} |\nabla^\varkappa u^*(x)|^2 \, dx. \end{aligned}$$

Finally by using the Pólya and Szegö rearrangement inequality (see for example [8,6]).

$$\|\nabla^\varkappa u^*\|_2 \le \|\nabla^\varkappa u\|_2, \qquad 0 \le \varkappa \le 1,$$

we complete the proof of Theorem 1.7.

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