

BEREZIN–LI–YAU INEQUALITIES ON DOMAINS ON THE SPHERE

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ABSTRACT. We prove Berezin–Li–Yau inequalities for the Dirichlet and Neumann eigenvalues on domains on the sphere \mathbb{S}^{d-1} . A sharp explicit bound for the sums of the Neumann eigenvalues is obtained for all dimensions d . In the case of \mathbb{S}^2 we also obtain sharp lower bounds for the vector Dirichlet Laplacian and the Stokes operator.

1. INTRODUCTION

Let Ω be a bounded domain in \mathbb{R}^d with volume $|\Omega|$. We denote by $\{\lambda_k\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$ the eigenvalues of the Laplacian $-\Delta$ with Dirichlet and Neumann boundary conditions, respectively. In the case of the Neumann boundary conditions we additionally assume that boundary $\partial\Omega$ is sufficiently regular so that the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact.

The following bounds hold for the Riesz means of λ_k and μ_k

$$\begin{aligned} \sum (\lambda - \lambda_k)_+^\sigma &\leq L_{\sigma,d}^{\text{cl}} |\Omega| \lambda^{\sigma+d/2}, \\ \sum (\lambda - \mu_k)_+^\sigma &\geq L_{\sigma,d}^{\text{cl}} |\Omega| \lambda^{\sigma+d/2}, \end{aligned} \tag{1.1}$$

where $x_+ = \max(0, x)$, $\sigma \geq 1$, and

$$L_{\sigma,d}^{\text{cl}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\sigma d\xi = \frac{\Gamma(\sigma + 1)}{(4\pi)^{2/d} \Gamma(\sigma + d/2 + 1)}. \tag{1.2}$$

The first one follows from more general results of [2]. A direct proof of both is given in [17].

A lower bound for the sums of the Dirichlet eigenvalues was obtained in [20]

$$\sum_{k=1}^n \lambda_k \geq \frac{d}{2+d} \left(\frac{(2\pi)^d}{\omega_d |\Omega|} \right)^{2/d} n^{1+2/d}, \tag{1.3}$$

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and a similar upper bound for the Neumann eigenvalues was proved in [16]

$$\sum_{k=1}^n \mu_k \leq \frac{d}{2+d} \left(\frac{(2\pi)^d}{\omega_d |\Omega|} \right)^{2/d} n^{1+2/d}. \quad (1.4)$$

We point out that the fact that the coefficients are constant is essential in the proof. Also, the constants in (1.1) and (1.3), (1.4) are sharp in view of the Weyl asymptotic formula for the eigenvalues. Finally, it was observed in [18] that inequalities (1.1) with $\sigma = 1$ and (1.3), (1.4) are pairwise equivalent, and the equivalence is realized by the Legendre transform and explains why they are called Berezin–Li–Yau inequalities.

Note that for domains $\Omega \subset \mathbb{R}^d$ there are many results where the authors obtained Berezin–Li–Yau inequalities with remainder terms. Among them are [7], [24] (with $\sigma \geq 3/2$) for the Berezin inequality and [6], [21], [14], [25], [26] for the Li–Yau estimate. Moreover, improved such inequalities for magnetic operators were recently proved in [15].

Our interest in this paper is in proving Berezin–Li–Yau inequalities for the Dirichlet and Neumann eigenvalues of the Laplace–Beltrami operator on a domain Ω on the unit sphere \mathbb{S}^{d-1} . Now Ω is a (curved) domain on \mathbb{S}^{d-1} with $(d-1)$ -dimensional surface area $|\Omega| \leq |\mathbb{S}^{d-1}| =: \sigma_d$. As before we denote by $\{\lambda_k\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$ the eigenvalues of the Laplace–Beltrami operator $-\Delta$ with Dirichlet and Neumann boundary conditions, respectively.

The main general results of Section 2 are the following bounds for the Riesz means of order $\sigma \geq 1$ of the eigenvalues $\{\lambda_k\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$:

$$\begin{aligned} \sum_{j=1}^{\infty} (\lambda - \lambda_j)_+^\sigma &\leq \frac{|\Omega|}{\sigma_d} \sum_{n=0}^{\infty} (\lambda - \Lambda_n)_+^\sigma k_d(n), \\ \sum_{j=1}^{\infty} (\lambda - \mu_j)_+^\sigma &\geq \frac{|\Omega|}{\sigma_d} \sum_{n=0}^{\infty} (\lambda - \Lambda_n)_+^\sigma k_d(n). \end{aligned} \quad (1.5)$$

Here Λ_n and $k_d(n)$ are the eigenvalues and their multiplicities of the Laplace operator on the whole sphere \mathbb{S}^{d-1} , see (1.10)–(1.11).

Next, we set $\sigma = 1$ so that the right-hand side, denoted below by $F_{\mathbb{S}^{d-1}}(\lambda)$, is a continuous piecewise linear function with change of slope at $\lambda = \Lambda_N$, $N = 0, 1, \dots$, and by means of the explicit expression for $F_{\mathbb{S}^{d-1}}(\Lambda_N)$ proved in Section 5 we show in Section 4 that in all dimensions

$$F_{\mathbb{S}^{d-1}}(\lambda) := \frac{|\Omega|}{\sigma_d} \sum_{n=0}^{\infty} (\lambda - \Lambda_n)_+ k_d(n) \geq L_{1,d-1}^{\text{cl}} |\Omega| \lambda^{1+(d-1)/2},$$

where the inequality is strict for all $\lambda > 0$ when $d - 1 \geq 3$, while in 2D case we have equality for $\lambda = \Lambda_N$. This gives that

$$\sum_{j=1}^{\infty} (\lambda - \mu_j)_+ \geq L_{1,d-1}^{\text{cl}} |\Omega| \lambda^{1+(d-1)/2}.$$

Applying the Legendre transform for both sides we obtain a Li-Yau-type upper bound for the sums of the first n eigenvalues of the Neumann Laplacian on $\Omega \subseteq \mathbb{S}^{d-1}$

$$\sum_{k=1}^n \mu_k \leq \frac{d-1}{2+d-1} \left(\frac{(2\pi)^{d-1}}{\omega_{d-1} |\Omega|} \right)^{2/(d-1)} n^{1+2/(d-1)}, \quad (1.6)$$

which looks exactly the same as in the case of the Euclidean space [16].

The situation with the Dirichlet eigenvalues is different and a Li-Yau estimate in the form (1.3) cannot hold, since the first eigenvalue on the whole sphere is 0. Therefore we restrict ourselves to the case of \mathbb{S}^2 , where we take advantage of the fact that $F_{\mathbb{S}^2}(\Lambda_N) = L_{1,2}^{\text{cl}} |\Omega| \Lambda_N^2$, and by evaluating the Legendre transform of $F_{\mathbb{S}^2}(\lambda)$ we obtain a Li-Yau-type lower bound:

$$\sum_{k=1}^n \lambda_k \geq \frac{2\pi}{|\Omega|} n \left(n - \frac{|\Omega|}{4\pi} \right), \quad (1.7)$$

which turns into the equality when $\Omega = \mathbb{S}^2$ and $n = N^2$, and which properly takes into account the behaviour of the first eigenvalue as $|\Omega| \rightarrow |\mathbb{S}^2| = 4\pi$:

$$\lambda_1 = \lambda_1(\Omega) \geq \frac{2\pi}{|\Omega|} \left(1 - \frac{|\Omega|}{4\pi} \right). \quad (1.8)$$

In this connection we observe that sharp estimates for the first eigenvalue of the Schrödinger operator on \mathbb{S}^{d-1} without the exclusion of the zero mode were obtained in [4].

The proof of the inequalities for Riesz means (1.5), on which the subsequent analysis is based, essentially relies on the pointwise identity (1.12) for the orthonormal spherical harmonics.

A similar pointwise identity holds for the gradients of the spherical harmonics (1.15) and makes it possible to prove Berezin-Li-Yau inequalities for the Dirichlet eigenvalues of the vector Laplacian. Unlike the scalar case, the vector Laplacian (we deal with the Laplace-de Rham operator) is strictly positive, since the sphere is simply connected. We consider only the two dimensional case, where a divergence free vector function have a scalar stream function, so that the identity (1.15) works equally well in the invariant spaces of potential and divergence free vector functions. We

obtain a Li–Yau-type inequality for the vector Dirichlet Laplacian, which looks exactly the same as in the case $\Omega \subset \mathbb{R}^2$. A more interesting example is the Stokes operator on a domain $\Omega \subseteq \mathbb{S}^2$:

$$\begin{aligned} -\Delta u_j + \nabla p_j &= \nu_j u_j, \\ \operatorname{div} u_j &= 0, \quad u_j|_{\partial\Omega} = 0, \end{aligned} \tag{1.9}$$

where the scalars p_j are the corresponding pressures. We show that

$$\sum_{k=1}^n \nu_k \geq \frac{2\pi}{|\Omega|} n^2.$$

We observe that this estimate is also exactly the same as the Li–Yau bound for the Stokes operator [12], [13] in the 2d case $\Omega \subset \mathbb{R}^2$.

We now recall the basic facts concerning the Laplace operator on the sphere \mathbb{S}^{d-1} . The one dimensional case $d = 2$ is somewhat special, so we assume below that $d \geq 3$. We have for the (scalar) Laplace–Beltrami operator $\Delta = \operatorname{div} \nabla$:

$$-\Delta Y_n^k = \Lambda_n Y_n^k, \quad k = 1, \dots, k_d(n), \quad n = 0, 1, 2, \dots \tag{1.10}$$

Here the Y_n^k are the orthonormal spherical harmonics which are chosen to be real-valued. Each eigenvalue $\Lambda_n = n(n + d - 2)$ has multiplicity

$$k_d(n) = \frac{1}{d-2} \binom{n+d-3}{d-3} (2n+d-2). \tag{1.11}$$

For example, for $d = 3$ we have for \mathbb{S}^2 $\Lambda_n = n(n + 1)$, $k_3(n) = 2n + 1$.

The following identity is essential in what follows [22]: for any $s \in \mathbb{S}^{d-1}$

$$\sum_{k=1}^{k_d(n)} Y_n^k(s)^2 = \frac{k_d(n)}{\sigma_d}, \tag{1.12}$$

where $|\mathbb{S}^{d-1}| =: \sigma_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of \mathbb{S}^{d-1} .

We note that identity (1.12) holds in a more general case of a compact homogeneous Riemannian manifold M [19]. Namely, if $\lambda \geq 0$ is an eigenvalue of $-\Delta$ on M with multiplicity $k_M(\lambda)$ and $y_k(x)$, $k = 1, \dots, k_M(\lambda)$ are the corresponding orthonormal eigenfunctions, then, for any $x \in M$,

$$\sum_{k=1}^{k_M(\lambda)} y_k(x)^2 = \frac{k_M(\lambda)}{|M|}, \tag{1.13}$$

where $|M| := \operatorname{vol}(M)$.

Based on this identity, lower bounds for the sums of the Dirichlet eigenvalues and upper bounds for the sums of the Neumann eigenvalues in an open

domain $\Omega \subseteq M$ were obtained in [23] in terms of the renormalized eigenvalue sum function. In the case of $\Omega \subseteq \mathbb{S}^2$ the results of [23] are written in the explicit form

$$\sum_{k=1}^n \lambda_k \geq \frac{2\pi}{|\Omega|} n \left(n - \frac{|\Omega|}{4\pi} \right), \quad \sum_{k=1}^n \mu_k \leq \frac{2\pi}{|\Omega|} n^2,$$

so that our estimate (1.7) and estimate (1.6) for $d - 1 = 2$ are not new. On the other hand, the explicit upper bound (1.6) for higher dimensions $d - 1 \geq 3$ is new.

We also observe that in the case of a homogeneous manifold M (with identity (1.13)) our estimates for the Riesz means (1.5) can be immediately written in the form

$$\begin{aligned} \sum_{j=1}^{\infty} (\lambda - \lambda_j)_+^{\sigma} &\leq \frac{|\Omega|}{|M|} \sum_{n=1}^{\infty} (\lambda - \tilde{\Lambda}_n)_+^{\sigma}, \\ \sum_{j=1}^{\infty} (\lambda - \mu_j)_+^{\sigma} &\geq \frac{|\Omega|}{|M|} \sum_{n=1}^{\infty} (\lambda - \tilde{\Lambda}_n)_+^{\sigma}, \end{aligned} \tag{1.14}$$

where $\{\tilde{\Lambda}_n\}_{n=1}^{\infty}$ are the eigenvalues (counting multiplicities) of $-\Delta$ on the whole M . We point out in this connection that the second inequality in (1.14) can be found for $\sigma = 1$ in [5]. We finally point out that universal estimates of the the $k + 1$ -th Dirichlet eigenvalue by the first k eigenvalues for a domain on the sphere were obtained in [3] (see also [8, 9, 10] and the references therein).

In the vector case we have the identity for the gradients of spherical harmonics that is similar to (1.12) (see [11]): for any $s \in \mathbb{S}^{d-1}$

$$\sum_{k=1}^{k_d(n)} |\nabla Y_n^k(s)|^2 = \Lambda_n \frac{k_d(n)}{\sigma_d}. \tag{1.15}$$

This identity is especially useful for inequalities for vector functions on \mathbb{S}^2 and we prove it for the sake of completeness. Substituting $\varphi(s) = Y_n^k(s)$ into the identity

$$\Delta \varphi^2 = 2\varphi \Delta \varphi + 2|\nabla \varphi|^2$$

we sum the results over $k = 1, \dots, k_d(n)$. In view of (1.12) the left-hand side vanishes and we obtain (1.15) since the $Y_n^k(s)$'s are the eigenfunctions corresponding to Λ_n .

2. INEQUALITIES FOR RIESZ MEANS OF THE DIRICHLET AND NEUMANN EIGENVALUES ON DOMAINS ON THE SPHERE

Let $\Omega \subseteq \mathbb{S}^{d-1}$ be a (curved) domain on \mathbb{S}^{d-1} with $(d-1)$ -dimensional surface measure $|\Omega| \leq \sigma_d$. We denote by $H^1(\Omega)$ the standard Sobolev space on Ω , and by $H_0^1(\Omega)$ its closed subspace of functions vanishing on $\partial\Omega$. Next, we define the Dirichlet Laplacian Δ_Ω^D and the Neumann Laplacian Δ_Ω^N via the quadratic forms with domains $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively.

We assume (in the case of the Neumann Laplacian) that the boundary $\partial\Omega$ is sufficiently regular so that the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact.

To fix notation we write the Dirichlet and Neumann eigenvalue problems

$$\begin{aligned} -\Delta\psi_j &= \lambda_j\psi_j, & \psi_j|_{\partial\Omega} &= 0, & 0 &\leq \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty, \\ -\Delta\omega_j &= \mu_j\omega_j, & \frac{\partial\omega_j}{\partial n}|_{\partial\Omega} &= 0, & 0 &= \mu_1 \leq \mu_2 \leq \dots \rightarrow +\infty, \end{aligned} \quad (2.1)$$

where n is the unit normal to $\partial\Omega$ tangent to \mathbb{S}^{d-1} . Here $\lambda_1 > 0$ if Ω is a proper domain on \mathbb{S}^{d-1} : $\text{meas}(\mathbb{S}^{d-1} \setminus \Omega) > 0$. If $\Omega = \mathbb{S}^{d-1}$, then $\lambda_j = \mu_j$ and the eigenvalues coincide with Λ_n with multiplicity (1.11). Both systems $\{\psi_j\}_{j=1}^\infty$ and $\{\omega_j\}_{j=1}^\infty$ are orthonormal in $L_2(\Omega)$.

The following result contains estimates for the Riesz means of the eigenvalues λ_j and μ_j .

Theorem 2.1. *Let $\sigma \geq 1$. Then inequalities (1.5) hold true.*

Proof. To simplify the notation we shall prove (1.5) for \mathbb{S}^2 , the general case is treated in exactly the same way. Furthermore, we start with $\sigma = 1$ and first consider the Dirichlet Laplacian.

We extend by zero each eigenfunction ψ_j to the whole \mathbb{S}^2 setting

$$\tilde{\psi}_j(s) = \begin{cases} \psi_j(s), & s \in \Omega; \\ 0, & s \notin \Omega. \end{cases}$$

and expand the result in spherical harmonics:

$$\tilde{\psi}_j(s) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} a_{jn}^k Y_n^k(s), \quad s \in \mathbb{S}^2,$$

where

$$a_{jn}^k = \int_{\mathbb{S}^2} \tilde{\psi}_j(s) Y_n^k(s) ds = \int_{\Omega} \psi_j(s) Y_n^k(s) ds.$$

Then using (1.10) and orthonormality of the Y_n^k 's we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} (\lambda - \lambda_j)_+ &= \sum_{j=1}^{\infty} \left(\int_{\Omega} (\lambda + \Delta) \psi_j \psi_j ds \right)_+ = \sum_{j=1}^{\infty} \left(\int_{\mathbb{S}^2} (\lambda + \Delta) \tilde{\psi}_j \tilde{\psi}_j ds \right)_+ \\ &= \sum_{j=1}^{\infty} \left(\int_{\mathbb{S}^2} (\lambda + \Delta) \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} a_{jn}^k Y_n^k(s) \sum_{n'=0}^{\infty} \sum_{k'=1}^{2n'+1} a_{jn'}^{k'} Y_{n'}^{k'}(s) ds \right)_+ \\ &= \sum_{j=1}^{\infty} \left(\sum_{n=0}^{\infty} (\lambda - n(n+1)) \sum_{k=1}^{2n+1} (a_{jn}^k)^2 \right)_+. \end{aligned}$$

We continue

$$\begin{aligned} \sum_{j=1}^{\infty} (\lambda - \lambda_j)_+ &\leq \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} (\lambda - n(n+1))_+ \sum_{k=1}^{2n+1} (a_{jn}^k)^2 = \\ &= \sum_{n=0}^{\infty} (\lambda - n(n+1))_+ \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} (a_{jn}^k)^2. \end{aligned}$$

We now consider the last double sum

$$\begin{aligned} \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} (a_{jn}^k)^2 &= \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} (\psi_j, Y_n^k)_{L_2(\Omega)}^2 = \sum_{k=1}^{2n+1} \int_{\Omega} Y_n^k(s)^2 ds = \\ &= \int_{\Omega} \sum_{k=1}^{2n+1} Y_n^k(s)^2 ds = \int_{\Omega} \frac{2n+1}{4\pi} ds = (2n+1) \frac{|\Omega|}{4\pi}, \end{aligned}$$

where the second equality is the Parseval identity for the expansion of a fixed function $Y_n^k(s)|_{s \in \Omega}$ in the Fourier series with respect to a complete orthonormal system $\{\psi_j\}_{j=1}^{\infty}$ in $L_2(\Omega)$, while the fourth equality is precisely (1.12).

As a result, we obtain

$$\sum_{j=1}^{\infty} (\lambda - \lambda_j)_+ \leq \frac{|\Omega|}{4\pi} \sum_{n=0}^{\infty} (\lambda - n(n+1))_+ (2n+1), \quad (2.2)$$

which is the first inequality in (1.5) for \mathbb{S}^2 and $\sigma = 1$.

We now consider the Neumann Laplacian. It is convenient to denote

$$\varphi_{\lambda}(t) := (\lambda - t)_+.$$

Then

$$\sum_{j=1}^{\infty} (\lambda - \mu_j)_+ = \sum_{j=1}^{\infty} \varphi_{\lambda}(\mu_j) = \sum_{j=1}^{\infty} \varphi_{\lambda}(\mu_j) \|\omega_j\|_{L_2(\Omega)}^2,$$

where ω_j are the orthonormal eigenfunctions defined in (2.1). We expand the ω_j 's in the spherical harmonics:

$$\omega_j(s) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} c_{jn}^k Y_n^k(s), \quad s \in \Omega, \quad \text{where } c_{jn}^k = \int_{\Omega} \omega_j(s) Y_n^k(s) ds,$$

so that

$$1 = \|\omega_j\|_{L_2(\Omega)}^2 = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} (c_{jn}^k)^2,$$

and as before

$$\sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} (c_{jn}^k)^2 = \int_{\Omega} \sum_{k=1}^{2n+1} Y_n^k(s)^2 ds = (2n+1) \frac{|\Omega|}{4\pi}.$$

Therefore setting $\sum_{k=1}^{2n+1} (c_{jn}^k)^2 =: b_{jn}$, we see that

$$\sum_{j=1}^{\infty} b_{jn} = (2n+1) \frac{|\Omega|}{4\pi} \quad \text{and} \quad \sum_{n=0}^{\infty} b_{jn} = 1.$$

We continue

$$\begin{aligned} \sum_{j=1}^{\infty} (\lambda - \mu_j)_+ &= \sum_{j=1}^{\infty} \varphi_{\lambda}(\mu_j) \sum_{n=0}^{\infty} b_{jn} = \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \varphi_{\lambda}(\mu_j) b_{jn} \frac{4\pi}{(2n+1)|\Omega|} \right) \frac{(2n+1)|\Omega|}{4\pi}. \end{aligned}$$

We consider the sum with respect to j . Since for each fixed n

$$\frac{4\pi}{(2n+1)|\Omega|} \sum_{j=1}^{\infty} b_{jn} = 1,$$

and the function φ_{λ} is convex, Jensen's inequality gives that

$$\sum_{j=1}^{\infty} \varphi_{\lambda}(\mu_j) b_{jn} \frac{4\pi}{(2n+1)|\Omega|} \geq \varphi_{\lambda} \left(\sum_{j=1}^{\infty} \mu_j b_{jn} \frac{4\pi}{(2n+1)|\Omega|} \right).$$

On the other hand,

$$\begin{aligned}
\sum_{j=1}^{\infty} \mu_j b_{jn} &= \sum_{j=1}^{\infty} \mu_j \sum_{k=1}^{2n+1} (c_{jn}^k)^2 = \sum_{j=1}^{\infty} \mu_j \sum_{k=1}^{2n+1} (\omega_j, Y_n^k)_{L_2(\Omega)}^2 = \\
&= \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} \mu_j (\omega_j, Y_n^k)_{L_2(\Omega)}^2 = n(n+1) \sum_{k=1}^{2n+1} \|Y_n^k\|_{L_2(\Omega)}^2 = \\
&= n(n+1) \int_{\Omega} \sum_{k=1}^{2n+1} Y_n^k(s)^2 ds = n(n+1) \frac{(2n+1)|\Omega|}{4\pi},
\end{aligned}$$

where the last equality is again precisely (1.12), and the fourth equality follows from the spectral theorem:

$$\begin{aligned}
\sum_{j=1}^{\infty} \mu_j (\omega_j, Y_n^k)_{L_2(\Omega)}^2 &= \left(\sum_{j=1}^{\infty} \mu_j (\omega_j, Y_n^k)_{L_2(\Omega)} \omega_j, Y_n^k \right)_{L_2(\Omega)} = \\
&= (-\Delta Y_n^k, Y_n^k)_{L_2(\Omega)} = n(n+1) \|Y_n^k\|_{L_2(\Omega)}^2.
\end{aligned}$$

Therefore

$$\varphi_{\lambda} \left(\sum_{j=1}^{\infty} \mu_j b_{jn} \frac{4\pi}{(2n+1)|\Omega|} \right) = \varphi_{\lambda}(n(n+1)) = ((\lambda - n(n+1))_+),$$

and we finally obtain

$$\sum_{j=1}^{\infty} (\lambda - \mu_j)_+ \geq \frac{|\Omega|}{4\pi} \sum_{n=0}^{\infty} (\lambda - n(n+1))_+ (2n+1), \quad (2.3)$$

which is the second inequality in (1.5) for \mathbb{S}^2 and $\sigma = 1$.

To complete the proof it remains to “lift” estimates (2.2), (2.3) to the powers $\sigma > 1$. This is done by using the argument in [1]. For any real number E evaluation of the integral gives the equality

$$E_+^{\sigma} = \frac{1}{c_{\sigma}} \int_0^{\infty} (E-t)_+ t^{\sigma-2} dt, \quad c_{\sigma} = B(2, \sigma-1). \quad (2.4)$$

Then in the Dirichlet case using (2.2) we have

$$\begin{aligned} \sum_{j=1}^{\infty} (\lambda - \lambda_j)_+^{\sigma} &= \frac{1}{c_{\sigma}} \int_0^{\infty} (\lambda - \lambda_j - t)_+ t^{\sigma-2} dt \leq \\ &\leq \frac{|\Omega|}{4\pi} \frac{1}{c_{\sigma}} \int_0^{\infty} \sum_{n=0}^{\infty} (\lambda - t - n(n+1))_+ (2n+1) t^{\sigma-2} dt = \\ &= \frac{|\Omega|}{4\pi} \sum_{n=0}^{\infty} (\lambda - n(n+1))_+^{\sigma} (2n+1), \end{aligned}$$

where we used (2.4) twice with $E = \lambda - \lambda_j$ and $E = \lambda - n(n+1)$.

We can do the same in the Neumann case, however, the direct proof in the case $\sigma = 1$ works for $\sigma > 1$, since the function $\varphi_{\lambda}(t)^{\sigma} = (\lambda - t)_+^{\sigma}$ is also convex. The proof is complete. \square

Remark 2.1. Inequalities (1.5) turn into equalities for $\Omega = \mathbb{S}^{d-1}$.

We set for $\sigma = 1$

$$F_{\mathbb{S}^{d-1}}(\lambda) := \frac{|\Omega|}{\sigma_d} \sum_{n=0}^{\infty} (\lambda - \Lambda_n)_+ k_d(n), \quad (2.5)$$

and give a more explicit expression for the function

$$f(\lambda) := \sum_{n=0}^{\infty} (\lambda - \Lambda_n)_+ k_d(n) \quad (2.6)$$

in estimates (1.5) in the case $\sigma = 1$.

Lemma 2.1. *The function $f(\lambda)$ is a piecewise linear function joining the points $[\Lambda_N, f(\Lambda_N)]$ in the plane $(\lambda, f(\lambda))$. For $\lambda = \Lambda_N$, $N = 0, 1, \dots$,*

$$f(\Lambda_N) = \sum_{n=0}^{N-1} k_d(n) (\Lambda_N - \Lambda_n). \quad (2.7)$$

Proof. The function $f(\lambda)$ is linear on every interval $\lambda \in [\Lambda_{N-1}, \Lambda_N]$. We have $f(\Lambda_0) = f(0) = 0$ and for $\lambda = \Lambda_N$ only the first N terms with $n = 0, \dots, N-1$ in (2.6) are non-zero, which gives (2.7). \square

3. THE 2D SPHERE \mathbb{S}^2

Lemma 3.1. *On \mathbb{S}^2 it holds*

$$f(\Lambda_N) = \frac{(N(N+1))^2}{2} = \frac{1}{2} \Lambda_N^2. \quad (3.1)$$

Proof. This is a direct calculation using (2.7) and that $\Lambda_n = n(n+1)$, $k_3(n) = 2n+1$. Alternatively, we may use the general formula (5.5) which works for all dimensions. \square

Corollary 3.1. *For all $\lambda \geq 0$*

$$F_{\mathbb{S}^2}(\lambda) := \frac{|\Omega|}{4\pi} \sum_{n=0}^{\infty} (\lambda - n(n+1))_+ (2n+1) \geq \frac{1}{8\pi} |\Omega| \lambda^2 = L_{1,2}^{\text{cl}} |\Omega| \lambda^2, \quad (3.2)$$

while

$$F_{\mathbb{S}^2}(\lambda) = \frac{1}{8\pi} |\Omega| \lambda^2 \quad \text{for } \lambda = \Lambda_N = N(N+1), \quad N = 0, 1, \dots \quad (3.3)$$

Proof. Equality (3.3) is just (3.1), and inequality (3.2) follows by convexity (see the top graph in Fig. 1). \square

Recalling estimate (1.5) for the Neumann eigenvalues μ_k we finally obtain the inequality

$$\sum_{j=1}^{\infty} (\lambda - \mu_j)_+ \geq \frac{1}{8\pi} |\Omega| \lambda^2.$$

We now take the Legendre transform of both sides of this inequality [18]. We recall that for a convex function $g(x)$ on \mathbb{R}_+ the Legendre transform $g^\vee(p)$ given by

$$g^\vee(p) := \sup_{x \geq 0} (px - g(x)).$$

The Legendre transform of the right-hand side is straightforward. For the left-hand side we have [18]

$$\left(\sum_{j=1}^{\infty} (\lambda - \mu_j)_+ \right)^\vee(p) = (p - [p]) \mu_{[p]+1} + \sum_{k=1}^{[p]} \mu_k. \quad (3.4)$$

If $g(x) \geq h(x)$, then $g^\vee(p) \leq h^\vee(p)$, and setting $p = n \in \mathbb{N}$ we obtain the following result.

Theorem 3.1. *Let Ω be an open domain on \mathbb{S}^2 with two-dimensional surface area $|\Omega|$ such that the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Then the sum of the first n eigenvalues μ_k of the Neumann Laplacian satisfies the estimate*

$$\sum_{k=1}^n \mu_k \leq \frac{2\pi}{|\Omega|} n^2. \quad (3.5)$$

Remark 3.1. We point out here that while the ordering of the numbers μ_j 's in the left-hand side in (3.4) does not play a role, the Legendre transform on the right-hand side automatically orders them in the non-decreasing way.

For the lower bound for the Dirichlet eigenvalues we need an explicit expression for the Legendre transform $F_{\mathbb{S}^2}^\vee(p)$ of the function $F_{\mathbb{S}^2}(\lambda)$.

Setting

$$\alpha := \frac{|\Omega|}{8\pi} \quad (3.6)$$

we first observe that for $\lambda \in [\Lambda_{N-1}, \Lambda_N]$

$$F_{\mathbb{S}^2}(\lambda) = \alpha((\Lambda_{N-1} + \Lambda_N)\lambda - \Lambda_{N-1}\Lambda_N) = \alpha(2N^2\lambda - N^2(N^2 - 1)). \quad (3.7)$$

In fact, in view of (3.3) we only have to verify that $F_{\mathbb{S}^2}(\lambda)$ in (3.7) satisfies $F_{\mathbb{S}^2}(\Lambda_{N-1}) = \alpha\Lambda_{N-1}^2$ and $F_{\mathbb{S}^2}(\Lambda_N) = \alpha\Lambda_N^2$. Next, the slope of the straight edge of the graph of $F_{\mathbb{S}^2}(\lambda)$ on each interval $\lambda \in [\Lambda_{N-1}, \Lambda_N]$ is equal to

$$\frac{\alpha(\Lambda_N^2 - \Lambda_{N-1}^2)}{\Lambda_N - \Lambda_{N-1}} = \alpha(\Lambda_N + \Lambda_{N-1}) = \alpha(N(N+1) + (N-1)N) = 2\alpha N^2.$$

We now find the expression for $F_{\mathbb{S}^2}^\vee(p)$. The function $F_{\mathbb{S}^2}^\vee(p)$ is a continuous monotone increasing piecewise linear function whose graph is a polygonal line with break points located at those points p where the line $p\lambda$ is parallel to the corresponding straight edge of the graph of the function $F_{\mathbb{S}^2}(\lambda)$. In other words, when the slope p is equal to $2\alpha N^2$. See Fig. 1.

Having said that we have

$$F_{\mathbb{S}^2}^\vee(p) = \max_{\lambda \in [\Lambda_{N-1}, \Lambda_N]} (p - \alpha(\Lambda_{N-1} + \Lambda_N))\lambda + \alpha\Lambda_{N-1}\Lambda_N,$$

$$p \in [\alpha(\Lambda_{N-2} + \Lambda_{N-1}), \alpha(\Lambda_{N-1} + \Lambda_N)] = [2\alpha(N-1)^2, 2\alpha N^2].$$

The coefficient of λ is negative, therefore λ has to be the smallest possible: $\lambda = \Lambda_{N-1}$. We finally obtain

$$F_{\mathbb{S}^2}^\vee(p) = N(N-1)(p - \alpha N(N-1)),$$

$$p \in [2\alpha(N-1)^2, 2\alpha N^2], \quad N = 1, 2, \dots$$

Observe that $F_{\mathbb{S}^2}^\vee(p) = 0$ for $p \in [0, 2\alpha]$.

We estimate $F_{\mathbb{S}^2}^\vee(p)$ from below. Let $p \in [2\alpha(N-1)^2, 2\alpha N^2]$, that is,

$$\sqrt{\frac{p}{2\alpha}} \leq N \leq \sqrt{\frac{p}{2\alpha}} + 1. \quad (3.8)$$

For a fixed p we now look at the function $F_{\mathbb{S}^2}^\vee(p) = N(N-1)(p - \alpha N(N-1))$, where N satisfies (3.8), as a quadratic parabola $x \rightarrow xp - \alpha x^2$ opening down.

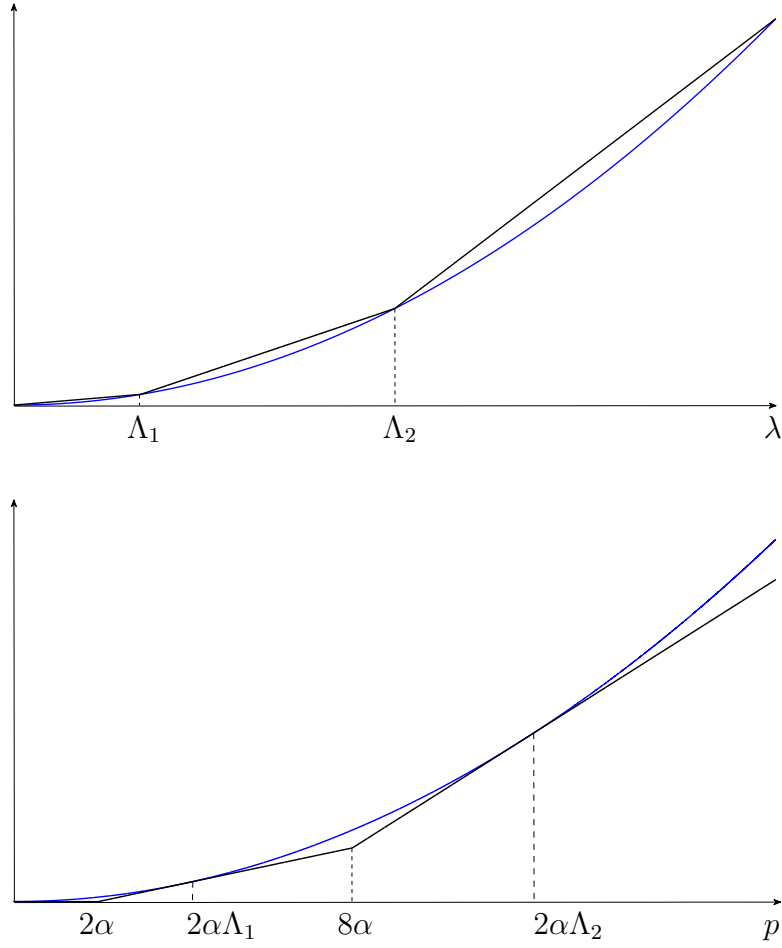


FIGURE 1. The graph of $\lambda \rightarrow \alpha\lambda^2$ and the polygonal graph of $F_{\mathbb{S}^2}(\lambda)$ are shown in the top figure with break points at $\lambda = \Lambda_1, \Lambda_2, \dots$. The Legendre transforms $(\alpha\lambda^2)^\vee(p) = p^2/(4\alpha)$ and $(F_{\mathbb{S}^2})^\vee(p)$ are shown in the bottom figure. The graph of $(F_{\mathbb{S}^2})^\vee(p)$ has break points at $p = 2\alpha N^2$ and is tangent to the curve $p^2/(4\alpha)$ at $p = 2\alpha\Lambda_N$, $N = 0, 1, \dots$.

Then

$$F_{\mathbb{S}^2}^\vee(p) \geq \min \left\{ N(N-1)(p - \alpha N(N-1)) \Big|_{N=\sqrt{\frac{p}{2\alpha}}}, \right. \\ \left. N(N-1)(p - \alpha N(N-1)) \Big|_{N=\sqrt{\frac{p}{2\alpha}+1}} \right\}.$$

However, both terms on the right-hand side are equal: for $u := \sqrt{p/(2\alpha)}$ and $\alpha = p/(2u^2)$ we have

$$u(u-1) \left(p - \frac{p}{2u^2} u(u-1) \right) = u(u+1) \left(p - \frac{p}{2u^2} u(u+1) \right) = \frac{p}{2}(u^2 - 1).$$

Therefore

$$F_{\mathbb{S}^2}^\vee(p) \geq \frac{p}{2}(u^2 - 1) = \frac{1}{4\alpha} p(p - 2\alpha) = \frac{2\pi}{|\Omega|} p \left(p - \frac{|\Omega|}{4\pi} \right).$$

Thus, we have proved the following lower bound for the sums of the Dirichlet eigenvalues.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{S}^2$ be an arbitrary domain. Then for any $n \geq 1$ the sum of the first n Dirichlet eigenvalues satisfies*

$$\sum_{k=1}^n \lambda_k \geq \frac{2\pi}{|\Omega|} n \left(n - \frac{|\Omega|}{4\pi} \right). \quad (3.9)$$

Corollary 3.2. *Since $n\lambda_n \geq \sum_{k=1}^n \lambda_k$, the lower bound (3.9) gives that for each $n \geq 1$*

$$\lambda_n \geq \frac{2\pi}{|\Omega|} \left(n - \frac{|\Omega|}{4\pi} \right). \quad (3.10)$$

Setting $n = 1$ we obtain

$$\lambda_1 = \lambda_1(\Omega) \geq \frac{2\pi}{|\Omega|} \left(1 - \frac{|\Omega|}{4\pi} \right). \quad (3.11)$$

The right-hand side vanishes as $|\Omega| \rightarrow 4\pi$ which is not surprising since on the whole sphere the first eigenvalue is zero.

Remark 3.2. If $\Omega = \mathbb{S}^2$ with $|\Omega| = 4\pi$, then the eigenvalues are $\Lambda_n = n(n+1)$ with multiplicities $2n+1$ starting from $n = 0$. Since

$$\sum_{n=0}^{N-1} (2n+1) = N^2 \quad \text{and} \quad \sum_{n=0}^{N-1} n(n+1)(2n+1) = \frac{1}{2}N^2(N^2 - 1),$$

it follows that inequality (3.9) turns into the equality for $n = N^2$.

Remark 3.3. In view of (3.3) the points $(\lambda, F_{\mathbb{S}^2}(\lambda))$ with $\lambda = \Lambda_N$ lie on the parabola $\alpha\lambda^2$. Therefore $F_{\mathbb{S}^2}^\vee(p) = (\alpha\lambda^2)^\vee(p)$ for $p = 2\alpha\Lambda_N = 2\alpha N(N+1)$. If $\alpha = 1/4$, then $p \in \mathbb{N}$ for all $N = 1, 2, \dots$. Therefore we have shown that

for $|\Omega| = 2\pi = |\mathbb{S}^2|/2$ and $n = N(N+1)/2$ the sum of the first n Dirichlet eigenvalues satisfies the lower bound

$$\sum_{k=1}^n \lambda_k \geq n^2, \quad \text{where } n = N(N+1)/2, \quad N = 1, 2, \dots$$

In particular, $\lambda_1 \geq 1$, while the universal estimate (3.11) gives $\lambda_1 \geq 1/2$.

The vector case. In the vector case we first define the Laplace operator acting on (tangent) vector fields on \mathbb{S}^2 as the Laplace–de Rham operator $-d\delta - \delta d$ identifying 1-forms and vectors. Then for a two-dimensional manifold (not necessarily \mathbb{S}^2) we have [11]

$$\Delta u = \nabla \operatorname{div} u - \operatorname{rot} \operatorname{rot} u, \quad (3.12)$$

where the operators $\nabla = \operatorname{grad}$ and div have the conventional meaning. The operator rot of a vector u is a scalar and for a scalar ψ , $\operatorname{rot} \psi$ is a vector:

$$\operatorname{rot} u := -\operatorname{div}(\mathbf{n} \times u) = \operatorname{div} u^\perp, \quad \operatorname{rot} \psi := -\mathbf{n} \times \nabla \psi = \nabla^\perp \psi,$$

where \mathbf{n} is the unit outward normal vector, so that in the local frame

$$-\mathbf{n} \times u = (u_2, -u_1) =: u^\perp.$$

We note that for a scalar ψ it holds

$$\operatorname{rot} \operatorname{rot} \psi = -\Delta \psi \quad (= -\operatorname{div} \operatorname{grad} \psi). \quad (3.13)$$

Integrating by parts, that is, using

$$(\nabla \psi, u)_{L_2(T\mathbb{S}^2)} = -(\psi, \operatorname{div} u)_{L_2(\mathbb{S}^2)}, \quad (\operatorname{rot} \psi, u)_{L_2(T\mathbb{S}^2)} = (\psi, \operatorname{rot} u)_{L_2(\mathbb{S}^2)},$$

we obtain

$$(-\Delta u, u)_{L_2(T\mathbb{S}^2)} = \|\operatorname{rot} u\|^2 + \|\operatorname{div} u\|^2. \quad (3.14)$$

The vector Laplacian has a complete in $L_2(T\mathbb{S}^2)$ orthonormal basis of vector-valued eigenfunctions. Using the notation

$$\{y_i\}_{i=1}^\infty = \{Y_n^1, \dots, Y_n^{2n+1}\}_{n=1}^\infty, \quad \{\lambda_i\}_{i=1}^\infty = \{\Lambda_n, \dots, \Lambda_n\}_{n=1}^\infty. \quad (3.15)$$

$2n+1$ times

we have

$$-\Delta w_j = \lambda_j w_j, \quad -\Delta v_j = \lambda_j v_j, \quad (3.16)$$

where

$$w_j = \lambda_j^{-1/2} \nabla^\perp y_j, \quad \operatorname{div} w_j = 0, \quad v_j = \lambda_j^{-1/2} \nabla y_j, \quad \operatorname{rot} v_j = 0.$$

Both (3.16), and the orthonormality of the w_j 's and v_j 's follow from (3.13), and (3.13) also implies the following commutation relations

$$\Delta \nabla = \nabla \Delta, \quad \Delta \nabla^\perp = \nabla^\perp \Delta,$$

which proves (3.16). For example, for the second relation we have

$$\Delta \nabla^\perp \psi = -\operatorname{rot} \operatorname{rot} \nabla^\perp \psi = -\operatorname{rot} \operatorname{rot} \operatorname{rot} \psi = \operatorname{rot} \Delta \psi = \nabla^\perp \Delta \psi. \quad (3.17)$$

We also point out that the fact that we are dealing with the sphere \mathbb{S}^2 does not play a role and (3.12) – (3.17) hold for any 2D manifold M .

Hence, on \mathbb{S}^2 , corresponding to the eigenvalue $\Lambda_n = n(n+1)$, where $n = 1, 2, \dots$, there are two families of $2n+1$ orthonormal vector-valued eigenfunctions $w_n^k(s)$ and $v_n^k(s)$, where $k = 1, \dots, 2n+1$ and (1.15) gives the following important identities: for any $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} |w_n^k(s)|^2 = \frac{2n+1}{4\pi}, \quad \sum_{k=1}^{2n+1} |v_n^k(s)|^2 = \frac{2n+1}{4\pi}. \quad (3.18)$$

We finally observe that since the sphere is simply connected, it follows that

$$\{\operatorname{div} u = 0, \operatorname{rot} u = 0\} \Rightarrow u = 0,$$

and therefore $-\Delta$ is strictly positive $-\Delta \geq \Lambda_1 I = 2I$. This fact explains why the Li–Yau bounds below look exactly the same as in the case of a bounded domain in \mathbb{R}^2 .

We now consider the Dirichlet eigenvalues in the vector case:

$$-\Delta u_j = \lambda_j u_j, \quad u_j|_{\partial\Omega} = 0,$$

where the vector-valued eigenfunctions u_j , $j = 1, \dots$ make up a complete orthonormal family in $L_2(\Omega, T\mathbb{S}^2)$.

Theorem 3.3. *Let $\Omega \subseteq \mathbb{S}^2$ be an arbitrary domain. Then for $\lambda \geq 0$*

$$\sum_{j=1}^{\infty} (\lambda - \lambda_j)_+ \leq 2 \cdot \frac{|\Omega|}{4\pi} \sum_{n=1}^{\infty} (\lambda - n(n+1))_+ (2n+1). \quad (3.19)$$

Proof. Before we proceed with the proof we point out the factor 2 on the right-hand side and the fact that the summation starts with $n = 1$.

The proof in turn repeats that of Theorem 2.1 and therefore will only be outlined. Let \tilde{u}_j be the extension by zero of u_j . We expand each \tilde{u}_j with respect to the orthonormal basis (3.16):

$$\tilde{u}_j(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{2n+1} (a_{jn}^k w_n^k(s) + c_{jn}^k v_n^k(s)), \quad s \in \mathbb{S}^2,$$

where

$$a_{jn}^k = (\tilde{u}_j(s), w_n^k)_{L_2(\Omega, T\mathbb{S}^2)}, \quad c_{jn}^k = (\tilde{u}_j(s), v_n^k)_{L_2(\Omega, T\mathbb{S}^2)}.$$

Then as in the proof of the Dirichlet case in Theorem 2.1 we obtain

$$\sum_{j=1}^{\infty} (\lambda - \lambda_j)_+ \leq \sum_{n=1}^{\infty} (\lambda - n(n+1))_+ \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} ((a_{jn}^k)^2 + (c_{jn}^k)^2).$$

For the double sum we use the identity (3.18) and find that

$$\begin{aligned} \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} ((a_{jn}^k)^2 + (c_{jn}^k)^2) &= \sum_{k=1}^{2n+1} \sum_{j=1}^{\infty} ((u_j, w_n^k)_{L_2(\Omega, T\mathbb{S}^2)}^2 + (u_j, v_n^k)_{L_2(\Omega, T\mathbb{S}^2)}^2) = \\ &= \sum_{k=1}^{2n+1} \int_{\Omega} (|w_n^k(s)|^2 + |v_n^k(s)|^2) ds = 2(2n+1) \frac{|\Omega|}{4\pi}. \end{aligned}$$

□

The following lemma shows that the omission of the zeroth term in the sum in (3.19) reverses the inequality in (3.2), see Fig. 2.

Lemma 3.2. *For all $\lambda \geq 0$*

$$F'_{\mathbb{S}^2}(\lambda) := \frac{|\Omega|}{4\pi} \sum_{n=1}^{\infty} (\lambda - n(n+1))_+ (2n+1) \leq \frac{1}{8\pi} |\Omega| \lambda^2 = L_{1,2}^{\text{cl}} |\Omega| \lambda^2. \quad (3.20)$$

Proof. Recalling definition (3.2) of the function $F_{\mathbb{S}^2}(\lambda)$ and the explicit expression for it (3.6), (3.7), what we have to prove is the following inequality

$$\alpha(2N^2\lambda - N^2(N^2 - 1)) - 2\alpha\lambda \leq \alpha\lambda^2, \quad \text{for } \lambda \in [\Lambda_{N-1}, \Lambda_N]$$

or $\lambda^2 - 2(N^2 - 1)\lambda + N^2(N^2 - 1) \geq 0$. However, this quadratic inequality holds for all λ , since its discriminant for $N \geq 1$ is negative:

$$-4(N^2 - 1) \leq 0.$$

□

Combining (3.19) and (3.20) we obtain that

$$\sum_{j=1}^{\infty} (\lambda - \lambda_j)_+ \leq 2L_{1,2}^{\text{cl}} |\Omega| \lambda^2. \quad (3.21)$$

Theorem 3.4. *Let $\Omega \subseteq \mathbb{S}^2$ be an arbitrary domain. Then for any $n \geq 1$ the sum of the first n Dirichlet eigenvalues of the vector Laplacian satisfies a Li-Yau-type lower bound*

$$\sum_{k=1}^n \lambda_k \geq \frac{\pi}{|\Omega|} n^2. \quad (3.22)$$

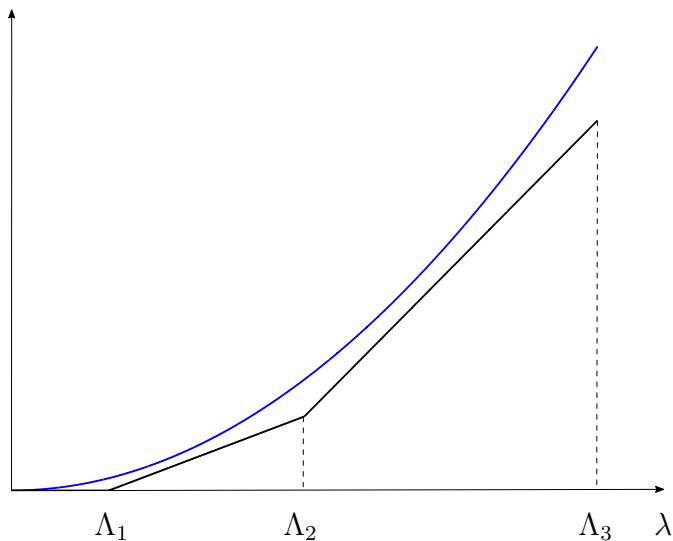


FIGURE 2. The polygonal graph of $F'_{\mathbb{S}^2}(\lambda) = F_{\mathbb{S}^2}(\lambda) - 2\alpha\lambda$ lies below the parabola $\lambda \rightarrow \alpha\lambda^2$.

Proof. Taking the Legendre transform of both sides of (3.21) we obtain (3.22). \square

We finally consider the case of the Stokes operator in a domain on \mathbb{S}^2 . Let ν_j and u_j be the eigenvalues and the divergence free vector-valued eigenfunctions of the Stokes operator defined by the quadratic form

$$u \rightarrow \int_{\Omega} (\operatorname{rot} u)^2 dS,$$

with domain

$$u \in H_0^1(\Omega, T\mathbb{S}^2), \quad \operatorname{div} u = 0.$$

If the boundary $\partial\Omega$ is sufficiently smooth, then this eigenvalue problem can be written in the form (1.9).

Theorem 3.5. *The sum of the first n eigenvalues of the Stokes operator on a domain $\Omega \subseteq \mathbb{S}^2$ satisfies a Li-Yau-type lower bound*

$$\sum_{k=1}^n \nu_k \geq \frac{2\pi}{|\Omega|} n^2. \quad (3.23)$$

Proof. Arguing as in the proof of Theorem 3.3 but this time using only the family of divergence free vector-valued eigenfunctions w_n^k (see (3.16), (3.18))

we obtain

$$\sum_{j=1}^{\infty} (\lambda - \nu_j)_+ \leq \frac{|\Omega|}{4\pi} \sum_{n=1}^{\infty} (\lambda - n(n+1))_+ (2n+1).$$

It remains to use (3.20) and apply the Legendre transform. \square

4. NEUMANN PROBLEM ON DOMAINS ON HIGHER DIMENSIONAL SPHERES

Proposition 4.1. *Let $d \geq 4$. For $\lambda > 0$ the function $F_{\mathbb{S}^{d-1}}(\lambda)$ defined in (2.5) satisfies the inequality*

$$F_{\mathbb{S}^{d-1}}(\lambda) > L_{1,d-1}^{\text{cl}} |\Omega| \lambda^{1+(d-1)/2}. \quad (4.1)$$

Proof. The proof essentially relies on the following general formula established in the Appendix:

$$f(\Lambda_N) := \sum_{n=0}^{N-1} k_d(n) (\Lambda_N - \Lambda_n) = \frac{(2N+d-1)(2N+d-3)}{d+1} \binom{N+d-2}{d-1}. \quad (4.2)$$

In view of the convexity of both functions in (4.1) and the polygonal shape of the graph of $F_{\mathbb{S}^{d-1}}(\lambda)$ it suffices to show that (4.1) holds for $\lambda = \Lambda_N = N(N+d-2)$ for all $N = 1, 2, \dots$. In other words, we have to show that

$$\frac{(2N+d-1)(2N+d-3)}{\sigma_d(d+1)} \binom{N+d-2}{d-1} > L_{1,d-1}^{\text{cl}} (N(N+d-2))^{(d+1)/2}.$$

The coefficient of the leading term N^{d+1} in the left-hand side is

$$\frac{4}{\sigma_d(d+1)\Gamma(d)} = L_{1,d-1}^{\text{cl}},$$

where the last equality follows from (1.2) and the duplication formula for the gamma function. Hence what we have to prove is the following inequality:

$$(N+(d-1)/2)(N+(d-3)/2)(N+d-2)(N+d-3) \cdots (N+1)N > [N(N+d-2)]^{(d+1)/2}. \quad (4.3)$$

As we have seen, in the two-dimensional case $d = 3$ we have equality here, but we are now dealing with the case $d \geq 4$.

The left-hand side contains $d+1$ factors. We set

$$N(N+d-2) = \lambda, \quad N = \sqrt{\lambda + (d-2)^2/4} - (d-2)/2.$$

Assume first that $d + 1$ is even. We combine the factors in (4.3) as follows (the first two terms are repeated twice: first as they are and the second time in the middle of the next group of factors)

$$\begin{aligned} N(N + d - 2) &= \lambda, \\ (N + 1)(N + d - 3) &= \lambda + d - 3 =: \lambda + \alpha_1, \\ (N + 2)(N + d - 4) &= \lambda + 2(d - 4) =: \lambda + \alpha_2, \\ &\vdots \\ (N + (d - 3)/2)(N + (d - 1)/2) &= \lambda + (d - 1)(d - 3)/4 =: \lambda + \alpha_{(d-1)/2}, \\ (N + (d - 1)/2)(N + (d - 3)/2) &= \lambda + (d - 1)(d - 3)/4 =: \lambda + \alpha_{(d-1)/2}, \end{aligned}$$

where all $\alpha_j > 0$. Now the inequality we are looking for becomes obvious

$$\lambda(\lambda + \alpha_{(d-1)/2}) \prod_{j=1}^{(d-3)/2} (\lambda + \alpha_j) > \lambda^{(d+1)/2}.$$

Finally, if $d + 1$ is odd, we act similarly, and the factor without pair is $N + (d - 2)/2$. But $N + (d - 2)/2 = \sqrt{\lambda + (d - 2)^2/4}$ and we obtain instead

$$\lambda \sqrt{\lambda + (d - 2)^2/4} \prod_{j=1}^{(d-2)/2} (\lambda + \alpha_j) > \lambda^{(d+1)/2}.$$

□

Theorem 4.1. *For any domain $\Omega \subseteq \mathbb{S}^{d-1}$ for which the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact the following lower bound holds for the Riesz means of order 1 of the Neumann eigenvalues*

$$\sum_{j=1}^{\infty} (\lambda - \mu_j)_+ \geq L_{1,d-1}^{\text{cl}} |\Omega| \lambda^{1+(d-1)/2}. \quad (4.4)$$

Proof. The proof is a combination of (1.5), (2.5) and (4.1). □

Theorem 4.2. *The sum of the first n eigenvalues μ_k of the Neumann Laplacian on a domain $\Omega \subseteq \mathbb{S}^{d-1}$, $d - 1 \geq 2$, satisfies the estimate*

$$\sum_{k=1}^n \mu_k \leq \frac{d-1}{d+1} \left(\frac{(2\pi)^{d-1}}{\omega_{d-1} |\Omega|} \right)^{2/(d-1)} n^{1+2/(d-1)}, \quad (4.5)$$

while each eigenvalue satisfies for $k = 0, 1, \dots$ the upper bound

$$\mu_{k+1} \leq \left(\frac{d+1}{2} \right)^{2/(d-1)} \left(\frac{(2\pi)^{d-1}}{\omega_{d-1} |\Omega|} \right)^{2/(d-1)} k^{2/(d-1)}. \quad (4.6)$$

Proof. Taking the Legendre transform of (4.4) we obtain (4.5). Next, we prove (4.6) following [17, Theorem 3.2]: since $\mu_1 = 0$, the counting function $N(\lambda, -\Delta_\Omega^N) := \#\{k, \mu_k < \lambda\}$ satisfies

$$N(\lambda, -\Delta_\Omega^N) \geq \frac{1}{\lambda} \sum_{j=1}^{\infty} (\lambda - \mu_j)_+.$$

Therefore, in view of (4.4),

$$N(\lambda, -\Delta_\Omega^N) \geq L_{1,d-1}^{\text{cl}} |\Omega| \lambda^{(d-1)/2} = \frac{2}{d+1} L_{0,d-1}^{\text{cl}} |\Omega| \lambda^{(d-1)/2},$$

which is equivalent to (4.6). \square

Remark 4.1. Upper bounds (4.5) and (4.6) are exactly the same as in the case $\Omega \subset \mathbb{R}^{d-1}$, see [16, 17].

5. APPENDIX. CALCULATION OF $f(\Lambda_N)$

We first recall the formula for the summation of the multiplicities (1.11)

$$\sum_{n=0}^N k_d(n) = k_{d+1}(N). \quad (5.1)$$

In fact, since

$$k_d(n) = \binom{d+n-1}{n} - \binom{d+n-3}{n-2},$$

(where the right hand side is the difference between the dimensions of the homogeneous polynomials of degrees n and $n-2$ in \mathbb{R}^d) the sum telescopes and we obtain (5.1).

We now consider

$$f(\Lambda_N) = \sum_{n=0}^{N-1} k_d(n) (\Lambda_N - \Lambda_n) = \Sigma_1(d, N) - \Sigma_2(d, N), \quad (5.2)$$

where

$$\Sigma_1(d, N) = \Lambda_N \sum_{n=0}^{N-1} k_d(n), \quad \Sigma_2(d, N) = \sum_{n=0}^{N-1} k_d(n) \Lambda_n.$$

For the first sum we use (5.1) and find that

$$\Sigma_1(d, N) = N(N+d-2)k_{d+1}(N-1) = N(2N+d-3) \binom{N+d-2}{d-1}. \quad (5.3)$$

Using (1.11) we write the second sum as follows

$$\begin{aligned}\Sigma_2(d, N) &= \frac{1}{d-2} \sum_{n=0}^{N-1} \binom{n+d-3}{d-3} (2n+d-2)n(n+d-2) = \\ &= \frac{1}{d-2} \sum_{n=0}^{N-1} \binom{n+d-3}{d-3} (2n^3 + 3(d-2)n^2 + (d-2)^2n).\end{aligned}$$

The last factor is a polynomial with respect to n of degree three without a constant term, and we represent it in the basis

$$\left\{ \frac{n}{d-2}, \frac{n(n-1)}{(d-2)(d-1)}, \frac{n(n-1)(n-2)}{(d-2)(d-1)d} \right\}$$

as follows

$$\begin{aligned}2n^3 + 3(d-2)n^2 + (d-2)^2n &= \\ = (d-2)(d-1)d \left(2 \frac{n(n-1)(n-2)}{(d-2)(d-1)d} + 3 \frac{n(n-1)}{(d-2)(d-1)} + \frac{n}{(d-2)} \right).\end{aligned}$$

Therefore

$$\begin{aligned}\Sigma_2(d, N) &= (d-1)d \sum_{n=0}^{N-1} \left[2 \binom{n+d-3}{d-3} \frac{n(n-1)(n-2)}{(d-2)(d-1)d} + \right. \\ &\quad \left. + 3 \binom{n+d-3}{d-3} \frac{n(n-1)}{(d-2)(d-1)} + \binom{n+d-3}{d-3} \frac{n}{d-2} \right] = \\ &= (d-1)d \sum_{n=0}^{N-1} \left[2 \binom{n+d-3}{d} + 3 \binom{n+d-3}{d-1} + \binom{n+d-3}{d-2} \right].\end{aligned}$$

By the well-known property that $\binom{m}{k} = \binom{m+1}{k+1} - \binom{m}{k+1}$ the three sums telescope and we obtain

$$\Sigma_2(d, N) = (d-1)d \left[2 \binom{N+d-3}{d+1} + 3 \binom{N+d-3}{d} + \binom{N+d-3}{d-1} \right].$$

In view of (5.3) we single out the factor $\binom{N+d-2}{d-1}$ here and after straight forward calculations we obtain

$$\Sigma_2(d, N) = \binom{N+d-2}{d-1} \frac{(d-1)(N-1)}{N+d-2} \left[\frac{2(N-2)(N-3)}{d+1} + 3(N-2) + d \right].$$

However, the last factor equals

$$\frac{(N+d-2)(2N+d-3)}{d+1},$$

which gives

$$\Sigma_2(d, N) = \frac{d-1}{d+1} \binom{N+d-2}{d-1} (N-1)(2N+d-3). \quad (5.4)$$

We finally obtain

$$f(\Lambda_N) = \Sigma_1(d, N) - \Sigma_2(d, N) = \frac{(2N+d-1)(2N+d-3)}{d+1} \binom{N+d-2}{d-1}, \quad (5.5)$$

which is (4.2).

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