

RESEARCH ARTICLE

Spectral enclosures and stability for non-self-adjoint discrete Schrödinger operators on the half-line

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Abstract

We make a spectral analysis of discrete Schrödinger operators on the half-line, subject to complex Robin-type boundary couplings and complex-valued potentials. First, optimal spectral enclosures are obtained for summable potentials. Second, general smallness conditions on the potentials guaranteeing a spectral stability are established. Third, a general identity which allows to generate optimal discrete Hardy inequalities for the discrete Dirichlet Laplacian on the half-line is proved.

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1 | INTRODUCTION

A huge number of papers have been devoted to effects of additive perturbation V to a given operator H_0 to the spectrum. With traditional motivations rooted in mathematical physics, H_0 is a differential operator, such as the Schrödinger or Dirac operator, and V stands for a scalar or matrix operator of multiplication. Typical questions of interests are: What is the location of the spectrum of the perturbed operator $H_0 + V$, if V belongs to a given class of potentials? How does the best possible spectral enclosure look like? Are some components of the spectrum preserved under suitable smallness or repulsive type conditions on the potentials? and so on.

If H_0 and V are self-adjoint, this type of problems have been intensively studied for more than a century, due to the needs of quantum mechanics, and the spectral properties are well understood.

On the other hand, the theory of non-self-adjoint V (or even H_0) is much less developed and the investigation is essentially restricted to the last two decades. However, there are new motivations (including quantum mechanics [2]), which make the analysis highly expedient and fashionable. It is also mathematically challenging because of the lack of tools based on the spectral theorem.

The new non-self-adjoint era of the aforementioned type of problems is certainly initiated by the highly influential work of Abramov, Aslanyan, and Davies from 2001 [1], where the authors derived an optimal spectral enclosure for Schrödinger operators on the line with complex-valued potentials. Ten years after, the optimal spectral bounds for the case of Schrödinger operators on the half-line were deduced by Frank, Laptev, and Seiringer in [13]. Instead of giving an incomplete list of works with similar goals in various settings, we rather refer to recent papers [15, 22] with a fairly large collection of references on the subject.

The discrete analogue of the celebrated result [1] has been established only recently in [16]. More specifically, the authors derived optimal spectral enclosures for discrete Schrödinger operators on \mathbb{Z} with complex-valued potentials in sequence spaces. The case of discrete Dirac operators on \mathbb{Z} was investigated in [4]. Except for these studies and few more relevant papers such as [9, 10, 14], we are not aware of more works on discrete counterparts of the (comparatively many) differential settings studied in the last two decades.

It is precisely the goal of the present paper to continue with the research project on spectral properties of non-self-adjoint discrete operators initiated in [4, 16]. Here we intend to present a discrete analogue of the continuous Schrödinger operators on the half-line studied in [13]. More specifically, in Theorem 3, we deduce a spectral enclosure for the discrete Schrödinger operator on \mathbb{N} , subject to a complex Robin ‘boundary condition’ and complex-valued ℓ^1 -potentials. The optimality is proven in Theorem 5. These results are presented in Section 3, while in Section 2 we introduce the discrete Robin Laplacian on \mathbb{N} as the unperturbed operator and analyze its spectral properties as a necessary preliminary.

In the rest of the present paper, we go beyond the setting of [13] by looking for conditions on the potentials guaranteeing the spectral stability of the discrete Robin Schrödinger operators. Here we adopt the notion of spectral stability as used in [15], meaning that the point, continuous and residual components of the spectra are preserved by the perturbations.

In this course, discrete Hardy inequalities enter the game. Therefore, Section 4 is devoted to Hardy inequalities for the discrete Robin Laplacians on \mathbb{N} . As the main result of this section, we prove in Theorem 10 a general identity that allows to generate optimal discrete Hardy inequalities and, in addition, identifies the remainder term in the inequality. As a concrete application, we obtain a one parameter family of optimal Hardy weights for the discrete Robin Laplacian on \mathbb{N} in Theorem 15. Although our initial motivation stems from the exploration of the spectral stability, results on the discrete Hardy inequalities of Section 4 are of independent interest.

In Theorems 17 and 22 of Section 5, we establish general conditions on the potential guaranteeing the spectral stability of the discrete Robin Schrödinger operator on \mathbb{N} . Here we restrict the otherwise complex Robin parameter to a real interval, making the Dirichlet and Neumann cases as two extreme cases. (The continuous analogue which is also new is established as Remark 21.) Finally, we combine these results with the discrete Hardy inequalities of Section 4 in order to deduce more explicit bounds on the potential implying the spectral stability in Theorem 25.

The paper is concluded by Section 6, in which we mention a challenging open problem, and by the Appendix with several illustrative and comparison plots.

2 | THE DISCRETE ROBIN LAPLACIANS ON \mathbb{N} AND THEIR SPECTRAL PROPERTIES

We start with several definitions to clarify what we mean by discrete analogues of Dirichlet, Neumann, and Robin Laplacians on $\mathbb{N} := \{1, 2, 3, \dots\}$.

First, the discrete differentiation is realized by the following backward and forward difference operators acting on $\ell^2(\mathbb{N})$:

$$(D\psi)_n := \begin{cases} \psi_{n-1} - \psi_n, & n > 1, \\ -\psi_1, & n = 1, \end{cases} \quad \text{and} \quad (D^*\psi)_n := \psi_{n+1} - \psi_n, \quad n \geq 1, \quad (1)$$

for $\psi \in \ell^2(\mathbb{N})$. We could have defined $(D\psi)_n := \psi_{n-1} - \psi_n$ for every $n \geq 1$ with the convention that $\psi_0 := 0$, which is the realization of the ‘Dirichlet condition’, while D^* satisfies no boundary condition, as in the continuous case. Then, analogically to the continuous setting, operators D^*D and DD^* represent the discrete Dirichlet and Neumann Laplacian on \mathbb{N} , respectively; see the well-known commutation scenario in the continuous case [23, p. 263].

The discrete Dirichlet and Neumann Laplacians are closely related to Jacobi operators J_0 and J_1 given by the tridiagonal matrices

$$J_0 := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad J_1 := \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with respect to the standard basis $\{e_n\}_{n \in \mathbb{N}}$ of $\ell^2(\mathbb{N})$. Indeed, one has

$$-\Delta_0 := D^*D = 2 - J_0 \quad \text{and} \quad -\Delta_1 := DD^* = 2 - J_1.$$

Since the relationship is through a mere constant shift and a sign change, spectral properties of the Laplacians are encoded in J_0 and J_1 . Therefore, with some abuse of terminology, we will refer to J_0 and J_1 as the discrete Dirichlet and Neumann Laplacians on \mathbb{N} , too.

In greater generality, we can introduce the one-parameter family of Jacobi operators

$$J_a := \begin{pmatrix} a & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a \in \mathbb{C},$$

and, in analogy to the above relations, also operators

$$-\Delta_a := 2 - J_a. \quad (2)$$

We refer to $-\Delta_a$ as well as J_a as the discrete Robin Laplacian on \mathbb{N} with the coupling constant $a \in \mathbb{C}$. Clearly, $-\Delta_0$ and $-\Delta_1$ are the discrete Dirichlet and Neumann Laplacian on \mathbb{N} , respectively.

While spectral properties of J_0 are well known, this seems to be not the case for J_a with general $a \in \mathbb{C}$. Therefore we summarize spectral properties of J_a in the next theorem. To this end, it is useful to introduce the Joukowski transform

$$z = z(k) := k + k^{-1}, \quad (3)$$

where $k \in \mathbb{C}$, $0 < |k| < 1$. The Joukowski transform $k \mapsto z(k)$ is a one-to-one mapping from the punctured open unit disk $\mathbb{D} \setminus \{0\}$ onto the set $\mathbb{C} \setminus [-2, 2]$ (the unit circle \mathbb{T} is mapped twice onto $[-2, 2]$).

Theorem 1. *For $a \in \mathbb{C}$, one has*

$$\sigma_c(J_a) = [-2, 2], \quad \sigma_r(J_a) = \emptyset, \quad \text{and} \quad \sigma_p(J_a) = \begin{cases} \emptyset & \text{if } |a| \leq 1, \\ \{a + a^{-1}\} & \text{if } |a| > 1. \end{cases}$$

Moreover, if $|a| > 1$, the only eigenvalue of J_a is simple, that is, of algebraic multiplicity 1. Further, for $z = k + k^{-1} \notin \sigma(J_a)$, where $0 < |k| < 1$, the Green kernel of J_a reads

$$(J_a - z)_{m,n}^{-1} = \frac{(k - a)k^{m+n-1} - (k^{-1} - a)k^{|n-m|+1}}{(1 - ak)(k - k^{-1})}, \quad m, n \in \mathbb{N}. \quad (4)$$

Proof. First, we analyze the point spectrum of J_a . For $k \in \mathbb{C}$, $0 < |k| \leq 1$, we seek the solution $\psi = \psi(k)$ of the eigenvalue equation

$$J_a \psi = (k + k^{-1})\psi,$$

which is determined uniquely up to a multiplicative constant. One readily verifies that the solution reads

$$\psi_n = \begin{cases} \psi_n(k) = (k - a)k^{n-1} - (k^{-1} - a)k^{-n+1} & \text{if } k \neq \pm 1, \\ \psi_n(\pm 1) = (\pm 1)^n(n \mp a(n-1)) & \text{if } k = \pm 1. \end{cases} \quad (5)$$

The only non-trivial square summable solution is $\psi = \psi(a^{-1}) = -\psi(a)$ provided that $|a| > 1$. It follows the assertion about the point spectrum $\sigma_p(J_a)$.

Second, we show that the resolvent set of J_a contains $\mathbb{C} \setminus [-2, 2]$ if $|a| \leq 1$, and $\mathbb{C} \setminus ([-2, 2] \cup \{a + a^{-1}\})$ if $|a| > 1$, that is, $\mathbb{C} \setminus ([-2, 2] \cup \sigma_p(J_a)) \subset \rho(J_a)$. The theory of Jacobi operators provides us with the general formula for the Green kernel

$$G_{m,n}(z) = \frac{1}{W(\psi, \varphi)} \times \begin{cases} \varphi_n \psi_m & \text{if } m \leq n, \\ \varphi_m \psi_n & \text{if } n < m, \end{cases}$$

where ψ is a solution of eigenvalue equation $J_a \psi = z\psi$ (which need not to belong to $\ell^2(\mathbb{N})$), $\varphi \in \ell^2(\mathbb{N})$ and fulfills $\varphi_{n-1} + \varphi_{n+1} = z\varphi_n$ for all $n > 1$, and the n -independent Wronskian $W(\psi, \varphi)$ is given by the formula

$$W(\psi, \varphi) = \psi_{n+1}\varphi_n - \psi_n\varphi_{n+1},$$

which is non-vanishing if and only if ψ and φ are linearly independent. In our case, writing (3), the sequence ψ is given by (5) and $\varphi_n = k^n$, $n \in \mathbb{N}$. Then

$$W(\psi, \varphi) = (1 - ak)(k - k^{-1})$$

and

$$G_{m,n}(z) = \frac{(k - a)k^{m+n-1} - (k^{-1} - a)k^{n-m+1}}{(1 - ak)(k - k^{-1})}, \quad \text{for } m \leq n,$$

and symmetrically in the case $m > n$. Note that, for $0 < |k| < 1$, $W(\psi, \varphi) = 0$ if and only if $k = a^{-1}$ and $|a| > 1$, which corresponds to the eigenvalue $a + a^{-1}$. Hence $G_{m,n}$ is well defined for all $z \in \mathbb{C} \setminus [-2, 2]$ if $|a| \leq 1$, and all $z \in \mathbb{C} \setminus ([-2, 2] \cup \{a + a^{-1}\})$ if $|a| > 1$.

Next, we verify that the operator $G(z)$ with matrix elements $G_{m,n}(z)$ is bounded whenever $0 < |k| < 1$ with the only exception of $k = a^{-1}$ in the case when $|a| > 1$. Then, taking also into account that $G(z)(J_a - z) = (J_a - z)G(z) = I$, as one readily checks, we will know that $\mathbb{C} \setminus ([-2, 2] \cup \sigma_p(J_a)) \subset \rho(J_a)$ and also formula (4) will be proven for all $z \in \mathbb{C} \setminus ([-2, 2] \cup \sigma_p(J_a))$.

Note that

$$G(z) = \alpha(k)H(k) + \beta(k)T(k),$$

where $H(k)$ is the Hankel matrix with entries $H_{m,n}(k) := k^{m+n-2}$, $T(k)$ is the Toeplitz matrix with entries $T_{m,n}(k) := k^{|m-n|}$, for $m, n \in \mathbb{N}$, and $\alpha(k)$ and $\beta(k)$ are constants. We show that both $H(k)$ and $T(k)$ are bounded for all $k \in \mathbb{D}$ and hence also $G(z)$ is bounded for all $z \in \mathbb{C} \setminus ([-2, 2] \cup \sigma_p(J_a))$. The Hankel operator $H(k)$ is actually Hilbert–Schmidt, since

$$\|H(k)\|_{\text{HS}} = \sqrt{\sum_{m,n=1}^{\infty} |H_{m,n}(k)|^2} = \sum_{n=0}^{\infty} |k|^{2n} = \frac{1}{1 - |k|^2} < \infty.$$

Using the general formula for the norm of a Toeplitz operator, see, for example, [3, § 2.8], we get

$$\|T(k)\| = \max_{\theta \in [-\pi, \pi]} \left| \sum_{n=-\infty}^{\infty} k^{|n|} e^{i\theta} \right| \leq \sum_{n=-\infty}^{\infty} |k|^{|n|} = \frac{1 + |k|}{1 - |k|} < \infty.$$

Third, we show that $[-2, 2] \subset \sigma(J_a)$. This follows from the Weyl theorem since

$$\lim_{n \rightarrow \infty} \frac{\|(J_a - 2 \cos \phi) \psi^{(n)}\|}{\|\psi^{(n)}\|} = 0,$$

where $\psi^{(n)} := (\psi_1, \dots, \psi_n, 0, 0, \dots)$ is truncated sequence (5) with $k = e^{i\phi}$ and $\phi \in [-\pi, \pi]$. Indeed, the limit can be readily computed using equations

$$\|(J_a - 2 \cos \phi) \psi^{(n)}\|^2 = |\psi_{n+1}|^2 \quad \text{and} \quad \|\psi^{(n)}\|^2 = \sum_{j=1}^n |\psi_j|^2$$

together with the formulas from (5).

To conclude the assertion about the spectrum of J_a and its parts, it is sufficient to verify that the residual spectrum is empty. By definition, $z \in \sigma_r(J_a)$ if and only if $z \notin \sigma_p(J_a)$ and $\text{Ran}(J_a - z)^\perp = \text{Ker}(J_a - z)^*$ is non-trivial. Since $J_a^* = J_{\bar{a}}$, the latter is equivalent to $z \in \sigma_p(J_a)$. Hence $\sigma_r(J_a) = (\mathbb{C} \setminus \sigma_p(J_a)) \cap \sigma_p(J_a) = \emptyset$.

In summary, the spectrum of J_a is purely continuous and given by the interval $\sigma_c(J_a) = [-2, 2]$, except for the existence of the unique discrete eigenvalue $\lambda := a + a^{-1}$ if $|a| > 1$. It remains to verify the simplicity of the eigenvalue. To this end, it suffices to check that the eigenvector $\psi(a^{-1})$ corresponding to the eigenvalue λ of J_a and the eigenvector $\psi(a^{-1})$ corresponding to the eigenvalue $\bar{\lambda}$ of the adjoint $J_a^* = J_{\bar{a}}$ are not orthogonal. This is straightforward since, by using (5), we obtain

$$\left\langle \overline{\psi(a^{-1})}, \psi(a^{-1}) \right\rangle = \sum_{n=1}^{\infty} \psi_n^2(a^{-1}) = (a^{-1} - a)^2 \sum_{n=1}^{\infty} a^{-2n+2} = a^2 - 1 \neq 0.$$

It completes the proof of the theorem. \square

Lastly, we mention a duality between J_a and J_{-a} which allows to restrict some parts of the forthcoming analysis to $a \geq 0$ without loss of generality. The proof is straightforward.

Proposition 2. *We have the equations*

$$UJ_aU^* = -J_{-a} \quad \text{and} \quad U(-\Delta_a)U^* = 4 + \Delta_{-a},$$

for the unitary operator $U := \text{diag}(1, -1, 1, -1, \dots)$ and all $a \in \mathbb{C}$.

3 | OPTIMAL SPECTRAL ENCLOSURES FOR J_a PERTURBED BY COMPLEX ℓ^1 -POTENTIALS

The goal of this section is to deduce optimal spectral bounds for the spectrum of the shifted operator $J_a + V$, where $V := \text{diag}(v_1, v_2, \dots)$ is a diagonal matrix with complex sequence $v = \{v_n\}_{n=1}^{\infty} \in \ell^1(\mathbb{N})$. We refer to the sequence v as well as the operator V as the potential.

For $z = k + k^{-1}$, where $|k| \leq 1$, we define the function

$$g_a(z) := \sup_{n \in \mathbb{N}} \left| 1 - \frac{k - a}{1 - ak} k^{2n-1} \right|. \quad (6)$$

If $|a| > 1$, then $g_a(a + a^{-1}) = \infty$.

Theorem 3. *Let $v \in \ell^1(\mathbb{N})$ and $a \in \mathbb{C}$. Then*

$$\sigma_p(J_a + V) \subset \{z \in \mathbb{C} \setminus (-2, 2) \mid \sqrt{|z^2 - 4|} \leq g_a(z) \|v\|_{\ell^1}\}. \quad (7)$$

Proof. It follows from the Birman–Schwinger principle, cf. [15, Corol. 4], that for any $z \in \mathbb{C}$ (including $z \in \sigma_c(J_a) = [-2, 2]$), we have the implication

$$\|K(z)\| < 1 \quad \Rightarrow \quad z \notin \sigma_p(J_a + V),$$

where $K(z) := |V|^{1/2}(J_0 - z)^{-1}|V|^{1/2} \operatorname{sgn} V$ is the Birman–Schwinger operator.

With the aid of the Cauchy–Schwarz inequality, we may estimate

$$\begin{aligned} \|K(z)u\|^2 &\leq \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \sqrt{|v_m|} \left| (J_a - z)^{-1}_{m,n} \sqrt{|v_n|} |u_n| \right| \right)^2 \leq \gamma_a^2(z) \sum_{m=1}^{\infty} |v_m| \left(\sum_{n=1}^{\infty} \sqrt{|v_n|} |u_n| \right)^2 \\ &\leq \gamma_a^2(z) \|v\|_{\ell^1}^2 \|u\|^2, \end{aligned}$$

for any $u \in \ell^2(\mathbb{N})$, where

$$\gamma_a(z) := \sup_{m,n \in \mathbb{N}} \left| (J_a - z)^{-1}_{m,n} \right|.$$

Note that the thresholds of the continuous spectrum ± 2 , as well as the discrete eigenvalue $a + a^{-1}$ if $|a| > 1$, are always included in the set on the right-hand side of (7). Next, using (4) for $z = k + k^{-1}$ with $|k| \leq 1$, where $k \neq \pm 1$ and $k \neq a^{-1}$ if $|a| > 1$, we obtain

$$\begin{aligned} \gamma_a(z) &= \sup_{\substack{n,m \in \mathbb{N} \\ m \leq n}} |k|^{n-m} \left| \frac{1 - ak - (k - a)k^{2m-1}}{(1 - ak)(k^{-1} - k)} \right| = \frac{1}{|k^{-1} - k|} \sup_{m \in \mathbb{N}} \left| 1 - \frac{k - a}{1 - ak} k^{2m-1} \right| \\ &= \frac{g_a(z)}{\sqrt{z^2 - 4}}. \end{aligned}$$

In total, we see that

$$\|K(z)\| \leq \gamma_a(z) \|v\|_{\ell^1} = \frac{g_a(z)}{\sqrt{z^2 - 4}} \|v\|_{\ell^1},$$

for $z \neq \pm 2$, and hence

$$\sigma_p(J_a + V) \subset \{z \in \mathbb{C} \mid \sqrt{|z^2 - 4|} \leq g_a(z) \|v\|_{\ell^1}\}.$$

Moreover, we can remove the interval $(-2, 2)$ from the spectral enclosure since it follows from the assumption $v \in \ell^1(\mathbb{N})$ that $\sigma_p(J_a + V) \cap (-2, 2) = \emptyset$. This is a well-known result based on an asymptotic behavior of the so-called Jost solutions of $J_0 + V$, see, for example, [24, Section 13.6]. More specifically, the reference deals with real $v \in \ell^1(\mathbb{N})$ only, however, the reality of v is not essential for the proof and the same assertion holds true for complex $v \in \ell^1(\mathbb{N})$, too (we are not aware of an exact reference for this result covering complex potentials). \square

To prove the optimality of (7), we will need the following auxiliary result.

Lemma 4. *For any $\kappa \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $|1 + \kappa k^{2n}| \geq 1$.*

Proof. Let $\kappa \in \mathbb{C}$ and $k \in \mathbb{C} \setminus \mathbb{R}$ be fixed. The claim holds trivially for $\kappa = 0$, therefore we further assume $\kappa \neq 0$. Then $|1 + \kappa k^{2n}| \geq 1$ if $\operatorname{Re}(\kappa k^{2n}) \geq 0$ for some $n \in \mathbb{N}$. This is true if there exists $n \in \mathbb{N}$

such that

$$\operatorname{Re} e^{i\phi+2ni\theta} \geq 0,$$

where $\phi := \arg \kappa$ and $\theta := \arg k$.

If $\theta \in \pi(\mathbb{R} \setminus \mathbb{Q})$, then the set $\{e^{i\phi+2ni\theta} \mid n \in \mathbb{N}\}$ is dense in the unit circle and hence the claim is obviously true. Next, suppose $\theta \in \pi\mathbb{Q}$. Note that $\theta \notin \pi\mathbb{Z}$ since $k \notin \mathbb{R}$ by assumptions. This means that the points from $\{e^{i\phi+2ni\theta} \mid n \in \mathbb{N}\}$ are vertices of a regular p -gon with $p \geq 2$. At least one of these vertices has to be located in the half-plane $\operatorname{Re} z \geq 0$. \square

Theorem 5. *The spectral bound (7) is optimal in the following sense: given $Q > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\sqrt{|z^2 - 4|} = g_a(z)Q$ there exist $n \in \mathbb{N}$ and $\omega \in \mathbb{C}$ with $|\omega| = Q$ such that z is the only eigenvalue of $J_a + \omega P_n$, where $P_n = \langle \cdot, e_n \rangle e_n$.*

Remark 6. Note that the optimality result does not apply to real boundary points of the spectral enclosure (7). It is clear that the boundary points belonging to the open interval $(-2, 2)$ cannot be eigenvalues. However, it remains an open problem whether the boundary points located in $\mathbb{R} \setminus [-2, 2]$ can be eigenvalues of $J_a + V$ for a potential V .

Proof of Theorem 5. Let $Q > 0$ and $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\sqrt{|z^2 - 4|} = g_a(z)Q$ be given. We consider the potential of the form $V = \omega P_n$ with $|\omega| = Q$ and determine $n \in \mathbb{N}$ as well as the argument of ω such that $z \in \sigma_p(J_a + \omega P_n)$. By (4), the Birman–Schwinger operator $K(z)$ corresponding to $V = \omega P_n$ is

$$K(z) = \frac{\omega}{k^{-1} - k} \left(1 - \frac{k - a}{1 - ak} k^{2n-1} \right) P_n.$$

It follows that $z = k + k^{-1}$, where $k \in \mathbb{D} \setminus (-1, 1)$ and $k \neq a^{-1}$, if $|a| > 1$, is an eigenvalue of $J_a + \omega P_n$, if and only if

$$\frac{\omega}{k^{-1} - k} \left(1 - \frac{k - a}{1 - ak} k^{2n-1} \right) = -1. \quad (8)$$

Note that the supremum in definition (6) of $g_a(z)$ is attained for some $n \in \mathbb{N}$ which follows from Lemma 4 and the fact that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{k - a}{1 - ak} k^{2n-1} \right) = 1.$$

Thus, we pick $n \in \mathbb{N}$ such that

$$g_a(z) = \left| 1 - \frac{k - a}{1 - ak} k^{2n-1} \right|.$$

It implies that moduli of both sides in (8) coincide. Now, it suffices to set

$$\arg \omega := \pi - \arg \left[\frac{1}{k^{-1} - k} \left(1 - \frac{k - a}{1 - ak} k^{2n-1} \right) \right]$$

and equation (8) is fulfilled. \square

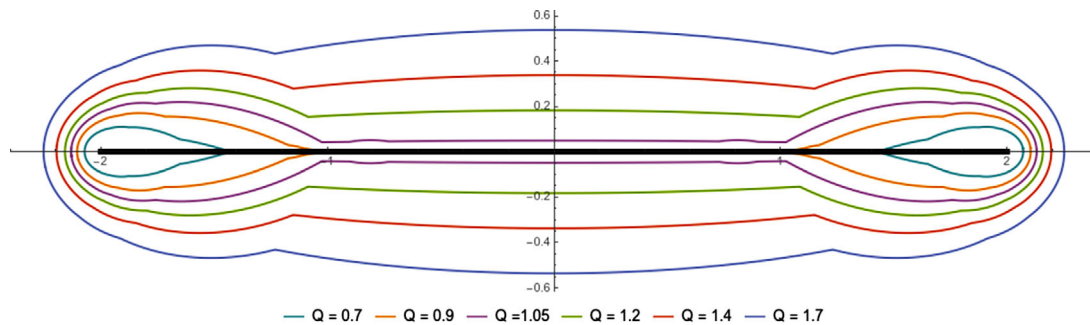


FIGURE 1 Spectral enclosures for the discrete Dirichlet Schrödinger operator ($a = 0$)

By estimating $g_a(z)$ from above in (7), one can obtain various non-optimal spectral enclosures whose advantage may be their simpler form. For example, one has

$$g_a(z) \leq 1 + \left| \frac{k-a}{1-ak} \right|,$$

which implies the following corollary.

Corollary 7. For $v \in \ell^1(\mathbb{N})$ and $a \in \mathbb{C}$, we have

$$\sigma_p(J_a + V) \subset \{\pm 2\} \cup \left\{ k + k^{-1} \mid 0 < |k| < 1, \left| k^{-1} - k \right| |1 - ak| \leq (|1 - ak| + |k - a|) \|v\|_{\ell^1} \right\}. \quad (9)$$

Recall that in the case of discrete Schrödinger operators on \mathbb{Z} with complex ℓ^1 -potentials, see [16, Theorem 1.1], the optimal spectral enclosure looks the same as (7) but the function g_a is not present. This is quite analogous to the known results in the continuous setting: While the optimal spectral enclosure of Schrödinger operators on the line with integrable complex potentials is a disk centered at the origin, see [1, Theorem 4], it is deformed in the case of Schrödinger operators on the half-line with Dirichlet boundary condition by an influence of an extra term similar to the function g_a , see [13, Theorem 1.1], and also [11].

Due to the presence of g_a in (9), the geometry of the spectral enclosure is highly non-trivial. Figures 1 and 2 show the boundary curves of the spectral enclosures for discrete Dirichlet and Neumann Schrödinger operators given by the equations

$$\sqrt{|z^2 - 4|} = g_a(z)Q, \quad a \in \{0, 1\},$$

for several values of the parameter $Q = \|v\|_{\ell^1}$, where

$$g_0(k^{-1} + k) = \sup_{n \in \mathbb{N}} |1 - k^{2n}| \quad \text{and} \quad g_1(k^{-1} + k) = \sup_{n \in \mathbb{N}} |1 + k^{2n-1}|.$$

Three more illustrative plots for other values of a are postponed to the Appendix.

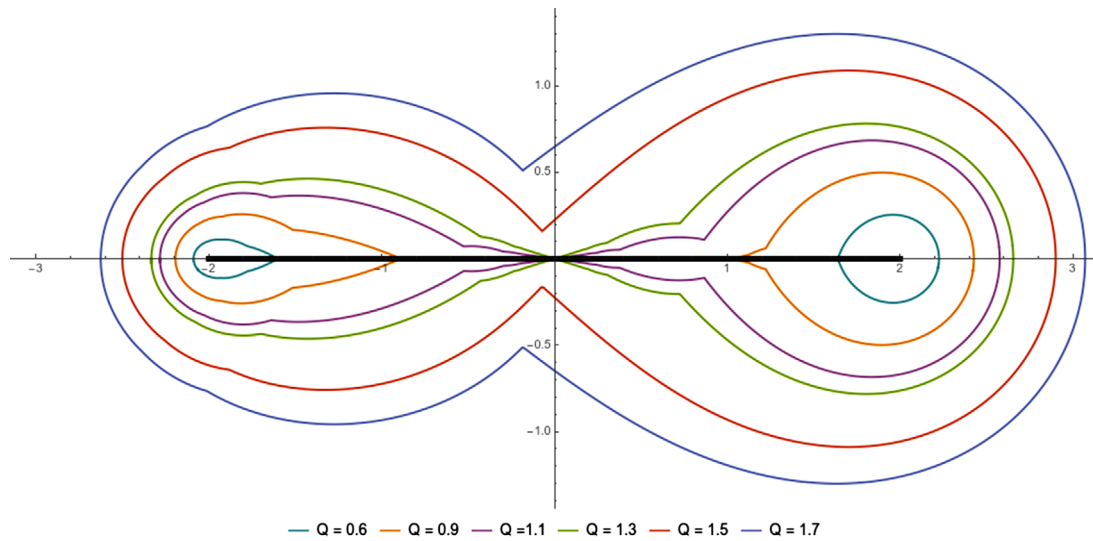


FIGURE 2 Spectral enclosures for the discrete Neumann Schrödinger operator ($a = 1$)

4 | INTERMEZZO: OPTIMAL DISCRETE HARDY INEQUALITIES FOR $-\Delta_a$

We interrupt the analysis of spectral properties of complex perturbations of discrete Laplacians by a study of discrete Hardy inequalities. These inequalities will be applied in the forthcoming section in results on a spectral stability of $-\Delta_a + V$ for small complex potentials V . Nevertheless, results of this section should be of independent interest.

Recall that, for $u \in \ell^2(\mathbb{N})$, the classical discrete Hardy inequality reads

$$\sum_{n=1}^{\infty} |u_{n-1} - u_n|^2 \geq \sum_{n=1}^{\infty} \frac{|u_n|^2}{4n^2}, \quad (10)$$

where $u_0 := 0$; see [21] for a historical account of the inequality. If we temporarily denote by W the diagonal operator defined by equations $W e_n := (4n)^{-2} e_n$, for $n \in \mathbb{N}$, and recall definition (2) with $a = 0$, we may rewrite (10) as the operator inequality

$$-\Delta_0 \geq W, \quad (11)$$

understood in the sense of quadratic forms. More generally, if (11) holds for some $0 \leq W = \text{diag}(w_1, w_2, \dots)$, the operator $-\Delta_0$ is said to satisfy a Hardy inequality and the sequence $w = \{w_n\}_{n=1}^{\infty}$ is said to be a Hardy weight.

A surprising fact observed by Keller, Pinchover, and Pogorzelski [18, 19] is that the classical discrete Hardy inequality (10) is not optimal in the following sense: A Hardy weight w is said to be an *optimal* Hardy weight, if for any other Hardy weight \tilde{w} , the point-wise inequality $\tilde{w} \geq w$ implies $\tilde{w} = w$. In [19], it is proved that the optimal Hardy weight actually reads

$$w_n := 2 - \sqrt{1 - \frac{1}{n}} - \sqrt{1 + \frac{1}{n}} > \frac{1}{4n^2}, \quad n \in \mathbb{N}. \quad (12)$$

At this point, the discrete and continuous case are not completely analogous since, in the continuous case, the well-known Hardy inequality

$$\int_0^\infty |u'(x)|^2 dx \geq \int_0^\infty \frac{|u(x)|^2}{4x^2} dx$$

valid for every $u \in W_0^{1,2}((0, \infty))$ is optimal; see [25, Sec. 8.1] and references therein.

Remark 8. In fact, a stronger notion of optimality, which involves two additional conditions on w , was considered in [19], based on the pioneering works [7, 8] in the continuous case. Namely, a Hardy weight w is said to be optimal, if and only if the following three conditions hold.

(opt1) If \tilde{w} is a Hardy weight such that $\tilde{w} \geq w$, then $\tilde{w} = w$.

(opt2) $\ker(-\Delta_0 - W) = \{0\}$.

(opt3) $(\forall \epsilon > 0)(\forall n \in \mathbb{N})(\exists \psi \in \ell^2(\mathbb{N}))(\|DQ_n \psi\|^2 < (1 + \epsilon)\langle \psi, WQ_n \psi \rangle)$,

where $W = \text{diag}(w_1, w_2, \dots)$, D is the difference operator defined in (1), and Q_n is the orthogonal projection onto $\text{span}\{e_n, e_{n+1}, \dots\}$. The Hardy weight (12) enjoys all the three properties (opt1–3).

Our next goal is to establish the discrete Hardy inequalities for the discrete Robin Laplacian, that is, to replace $-\Delta_0$ in (11) by $-\Delta_a$. At this point, we restrict the parameter a to the interval $[0, 1]$, so the Dirichlet and the Neumann cases correspond to the two extreme points 0 and 1, respectively. This restriction is without loss of generality, because the concept of Hardy inequalities is meaningful for the self-adjoint setting $a \in \mathbb{R}$ only, the couplings $a \notin [-1, 1]$ are disregarded because of the existence of a discrete eigenvalue (see Theorem 1) and the case of $a \in [-1, 0]$ can be deduced from $a \in [0, 1]$ by the duality of Proposition 2.

In fact, it makes sense to consider only $a \in [0, 1]$ since, in the Neumann case, there is no Hardy inequality analogously to the well-known continuous case. In other words, the operator $-\Delta_1$ is critical in the sense of the following statement.

Theorem 9. If $W \geq 0$ is a diagonal operator such that $-\Delta_1 \geq W$, then $W = 0$.

Proof. Suppose that the inequality $-\Delta_1 \geq W$ holds for some $W \geq 0$. We show that it implies $W = 0$. The idea is based on the observation that the constant sequence $\psi \equiv 1$ is annihilated by $-\Delta_1$. Since $1 \notin \ell^2(\mathbb{N})$, we need to use a regularized sequence which can be chosen, for instance, as follows:

$$\psi_n^{(N)} := \begin{cases} 1, & 1 \leq n < N, \\ \frac{2N-n}{N}, & N \leq n \leq 2N, \\ 0, & 2N < n, \end{cases}$$

for $n, N \in \mathbb{N}$. Note that $\psi^{(N)} \rightarrow 1$ point-wise as $N \rightarrow \infty$ and $\psi^{(N)} \leq \psi^{(N+1)}$ for all $N \in \mathbb{N}$. Further, an easy calculation shows that

$$\langle \psi^{(N)}, -\Delta_1 \psi^{(N)} \rangle = \|D^* \psi^{(N)}\|^2 = \sum_{n=1}^{\infty} |\psi_{n+1}^{(N)} - \psi_n^{(N)}|^2 = \frac{1}{N} \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

while

$$\left\langle \psi^{(N)}, W\psi^{(N)} \right\rangle = \sum_{n=1}^{\infty} w_n |\psi_n^{(N)}|^2 \rightarrow \sum_{n=1}^{\infty} w_n \quad \text{as } N \rightarrow \infty,$$

by the Monotone Convergence Theorem. It follows from the assumption $-\Delta_1 \geq W$ that

$$\left\langle \psi^{(N)}, -\Delta_1 \psi^{(N)} \right\rangle \geq \left\langle \psi^{(N)}, W\psi^{(N)} \right\rangle, \quad \forall N \in \mathbb{N}.$$

By sending $N \rightarrow \infty$, we obtain inequality $\sum_{n=1}^{\infty} w_n \leq 0$. Since $w_n \geq 0$ for all $n \in \mathbb{N}$, we conclude that $w_n = 0$ for all $n \in \mathbb{N}$. \square

Since the discrete Robin Laplacian $-\Delta_a$ is a rank-one perturbation of $-\Delta_0$, namely

$$-\Delta_a = -\Delta_0 - aP_1, \quad (13)$$

where $P_1 := \langle \cdot, e_1 \rangle e_1$ as in Theorem 5, we can deduce Hardy inequalities for $-\Delta_a$ from those for $-\Delta_0$.

Next, we prove an abstract identity which yields a method for generating Hardy inequalities for the discrete Dirichlet Laplacian. Moreover, it determines the remainder term in the Hardy inequality and, via a control of the remainder, provides a sufficient condition guaranteeing the optimality. Recall that here we mean by the optimality the property (opt1) from Remark 8. Spaces of finitely supported sequences indexed by \mathbb{N} and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are denoted by $C_0(\mathbb{N})$ and $C_0(\mathbb{N}_0)$, respectively.

Theorem 10. *Let $g = \{g_n\}_{n=1}^{\infty}$ be a positive sequence such that $-\Delta_0 g \geq 0$ entrywise. Then, for any $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$, we have the identity*

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 = \sum_{n=1}^{\infty} w_n |u_n|^2 + \sum_{n=2}^{\infty} \left| \sqrt{\frac{g_{n-1}}{g_n}} u_n - \sqrt{\frac{g_n}{g_{n-1}}} u_{n-1} \right|^2, \quad (14)$$

where

$$w_n := \frac{(-\Delta_0 g)_n}{g_n}.$$

In particular, we obtain the generalized discrete Hardy inequality

$$\sum_{n=1}^{\infty} w_n |u_n|^2 \leq \sum_{n=1}^{\infty} |u_n - u_{n-1}|^2.$$

Moreover, if there exists a sequence of elements $\xi^N \in C_0(\mathbb{N})$ such that $\xi^N \leq \xi^{N+1}$, $\xi^N \rightarrow 1$ as $N \rightarrow \infty$ pointwise, and

$$\lim_{N \rightarrow \infty} \sum_{n=2}^{\infty} g_n g_{n-1} \left| \xi_n^N - \xi_{n-1}^N \right|^2 = 0, \quad (15)$$

then w is optimal.

Proof. Let us temporarily denote $h := -Dg$. Then for any $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$, one has

$$\left| \sqrt{1 - \frac{h_n}{g_n}} u_n - \sqrt{1 + \frac{h_n}{g_{n-1}}} u_{n-1} \right|^2 = \left(1 - \frac{h_n}{g_n}\right) |u_n|^2 + \left(1 + \frac{h_n}{g_{n-1}}\right) |u_{n-1}|^2 - 2 \operatorname{Re}(\bar{u}_n u_{n-1}),$$

which further implies the identity

$$\left| \sqrt{\frac{g_{n-1}}{g_n}} u_n - \sqrt{\frac{g_n}{g_{n-1}}} u_{n-1} \right|^2 = |u_n - u_{n-1}|^2 - h_n \left(\frac{|u_n|^2}{g_n} - \frac{|u_{n-1}|^2}{g_{n-1}} \right),$$

for all $n \in \mathbb{N}$, where the terms

$$\sqrt{\frac{g_n}{g_{n-1}}} u_{n-1} \quad \text{and} \quad \frac{|u_{n-1}|^2}{g_{n-1}}$$

are to be interpreted as zeros for $n = 1$. Then by summing by parts, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} w_n |u_n|^2 &= \sum_{n=1}^{\infty} (h_n - h_{n+1}) \frac{|u_n|^2}{g_n} = \sum_{n=1}^{\infty} h_n \left(\frac{|u_n|^2}{g_n} - \frac{|u_{n-1}|^2}{g_{n-1}} \right) \\ &= \sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 - \sum_{n=1}^{\infty} \left| \sqrt{\frac{g_{n-1}}{g_n}} u_n - \sqrt{\frac{g_n}{g_{n-1}}} u_{n-1} \right|^2, \end{aligned}$$

which establishes (14) since the first term of the last sum vanishes.

Next, let a sequence of elements $\xi^N \in C_0(\mathbb{N})$ satisfying the assumptions is given. Suppose further that \tilde{w} is another Hardy weight, that is,

$$\sum_{n=1}^{\infty} \tilde{w}_n |u_n|^2 \leq \sum_{n=1}^{\infty} |u_n - u_{n-1}|^2,$$

for all $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$, and $\tilde{w} \geq w$. Then, by (14), we have

$$0 \leq \sum_{n=1}^{\infty} (\tilde{w}_n - w_n) |u_n|^2 \leq \sum_{n=2}^{\infty} \left| \sqrt{\frac{g_{n-1}}{g_n}} u_n - \sqrt{\frac{g_n}{g_{n-1}}} u_{n-1} \right|^2$$

for all $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$. Plugging $u_n := g_n \xi_n^N$ into the last expression, we get

$$0 \leq \sum_{n=1}^{\infty} (\tilde{w}_n - w_n) \left| g_n \xi_n^N \right|^2 \leq \sum_{n=2}^{\infty} g_n g_{n-1} \left| \xi_n^N - \xi_{n-1}^N \right|^2.$$

Using (15) and the assumptions $\xi^N \leq \xi^{N+1} \rightarrow 1$, we arrive, by the Monotone Convergence Theorem, at the equality

$$\sum_{n=1}^{\infty} (\tilde{w}_n - w_n) g_n^2 = 0.$$

Since each term of the sum is non-negative and $g_n \neq 0$ for all $n \in \mathbb{N}$, we conclude that $w = \tilde{w}$. \square

Remark 11. Theorem 10 is a generalization of [20, Theorem 1].

An interesting corollary of Theorem 10 is that we have an infinitely many fully explicit optimal discrete Hardy inequalities.

Corollary 12. *For any $q \in (0, 1/2]$, we have the optimal discrete Hardy inequality*

$$\sum_{n=1}^{\infty} w_n(q) |u_n|^2 \leq \sum_{n=1}^{\infty} |u_n - u_{n-1}|^2,$$

where

$$w_n(q) := 2 - \left(1 - \frac{1}{n}\right)^q - \left(1 + \frac{1}{n}\right)^q = \begin{cases} 2 - 2^q & \text{if } n = 1, \\ 2q \sum_{k=1}^{\infty} \frac{(1-q)_{2k-1}}{(2k)!} \frac{1}{n^{2k}} & \text{if } n \geq 2. \end{cases}$$

Remark 13. Note that, for $q \in (0, 1/2)$, one has

$$w_1(q) > w_1\left(\frac{1}{2}\right), \quad \text{while} \quad w_n(q) < w_n\left(\frac{1}{2}\right), \quad \text{for all } n \text{ sufficiently large.}$$

Curiously, for $n \geq 2$, the maximum of w_n in $(0, 1/2]$ is attained at the point

$$q_n = \frac{\log(-\log(1 - 1/n)) - \log(\log(1 + 1/n))}{\log(1 + 1/n) - \log(1 - 1/n)} = 1 - \frac{1}{24n^2} + O\left(\frac{1}{n^4}\right), \quad \text{as } n \rightarrow \infty.$$

If $q \in (1/2, 1)$, the discrete Hardy inequality with the weight $w_n(q)$ still holds but it is not optimal since $w_n(q) < w_n(1/2)$ for all $n \in \mathbb{N}$. For $q > 1$, the weight $w_n(q)$ is not non-negative, and $w(1)$ is trivial.

Proof of Corollary 12. To obtain the inequality, it suffices to put $g_n := n^q$ in Theorem 10.

To prove the optimality, we define

$$\xi_n^N := \begin{cases} 1 & \text{if } n < N, \\ \frac{2 \log N - \log n}{\log N} & \text{if } N \leq n \leq N^2, \\ 0 & \text{if } n > N^2, \end{cases}$$

for $N \geq 2$. Then $\xi^N \in C_0(\mathbb{N})$ such that $\xi^N \leq \xi^{N+1}$, $\xi^N \rightarrow 1$ as $N \rightarrow \infty$. Hence it remains to check condition (15). We have

$$\begin{aligned} \sum_{n=2}^{\infty} n^q (n-1)^q \left| \xi_n^N - \xi_{n-1}^N \right|^2 &= \frac{1}{\log^2 N} \sum_{n=N+1}^{N^2} n^q (n-1)^q \log^2 \left(\frac{n}{n-1} \right) \\ &\leq \frac{1}{\log^2 N} \sum_{n=N+1}^{N^2} \frac{n^q (n-1)^q}{(n-1)^2} \leq \frac{2}{\log^2 N} \sum_{n=N+1}^{N^2} \frac{1}{(n-1)^{2-2q}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{\log^2 N} \sum_{n=N+1}^{N^2} \frac{1}{(n-1)} \leq \frac{2}{\log^2 N} \int_N^{N^2} \frac{dn}{n-1} \\
&= \frac{2 \log(N+1)}{\log^2 N} \leq \frac{4}{\log N}.
\end{aligned}$$

Since the last expression tends to zero as $N \rightarrow \infty$, we are done. \square

Remark 14. Clearly, if $q = 1/2$, Corollary 12 reestablishes the result of [18] (see also [20]). Concerning the stronger notion of optimality from Remark 8, for $q \in (0, 1/2)$, one only has (opt1) and (opt2) but not (opt3).

Now we may combine Corollary 12 with (13) to deduce optimal discrete Hardy inequalities for the Robin Laplacian $-\Delta_a$. However, depending on the value of $a \in [0, 1)$, the range for the parameter $q \in (0, 1/2)$ has to be additionally restricted in order that the corresponding Hardy weight remains non-negative. To this end, for $a \in [0, 1)$, we denote

$$q_a := \min \left(\log_2(2-a), \frac{1}{2} \right). \quad (16)$$

Theorem 15 (discrete Hardy inequalities for $-\Delta_a$). *For any $a \in [0, 1)$ and $q \in (0, q_a]$, the Hardy inequality*

$$-\Delta_a \geq W$$

holds, where $W = \text{diag}(w_1, w_2, \dots)$ with

$$w_n = 2 - \left(1 - \frac{1}{n}\right)^q - \left(1 + \frac{1}{n}\right)^q - a\delta_{n,1}.$$

Moreover, all these Hardy weights are optimal.

5 | THE SPECTRAL STABILITY OF $-\Delta_a$ PERTURBED BY SMALL COMPLEX POTENTIALS

In this section, we provide sufficient conditions on the complex potential V that guarantee emptiness of either the point or the discrete spectrum of $J_a + V$ for $a \in [0, 1)$.

As an auxiliary result, we need to maximize the function

$$g_m(\theta; a) := \frac{|\sin(m\theta) - a \sin((m-1)\theta)|}{|\sin(\theta)| \sqrt{1 + a^2 - 2a \cos \theta}}, \quad (17)$$

for $\theta \in [-\pi, \pi]$, where $a \in [0, 1)$ and $m \in \mathbb{N}$ are parameters. The value $g_m(0, a)$ is conventionally defined as the continuous extension of $\theta \mapsto g_m(\theta, a)$ initially defined on $[-\pi, 0) \cup (0, \pi]$.

Lemma 16. For any $a \in [0, 1)$ and $m \in \mathbb{N}$, one has

$$\max_{\theta \in [-\pi, \pi]} g_m(\theta; a) = g_m(0; a) = \frac{m - a(m-1)}{1-a}.$$

Proof. Note the function $g_m(\cdot; a)$ is even, hence the analysis can be restricted to $[0, \pi]$. We will need the following two inequalities.

First, for all $\theta \in (0, \pi)$ and $m \in \mathbb{N}$, we have

$$\frac{|\sin(m\theta)|}{\sin(\theta)} \leq m. \quad (18)$$

Inequality (18) can be proven by induction in $m \in \mathbb{N}$. The induction step makes use of the inequality

$$\begin{aligned} \frac{|\sin((m+1)\theta)|}{\sin(\theta)} &= \frac{|\sin(m\theta)\cos(\theta) + \cos(m\theta)\sin(\theta)|}{\sin(\theta)} \leq \frac{|\sin(m\theta)|}{\sin(\theta)} |\cos(\theta)| + |\cos(m\theta)| \\ &\leq \frac{|\sin(m\theta)|}{\sin(\theta)} + 1, \end{aligned}$$

which holds true for all $\theta \in (0, \pi)$.

Second, for all $\theta \in (0, \pi)$, $a \in [0, 1)$ and $m \in \mathbb{N}$, we have

$$f_m(\theta; a) := \frac{m(1 - a \cos \theta) + a}{m(1 - a) + a} \leq \frac{1 - a \cos \theta}{1 - a} = \lim_{m \rightarrow \infty} f_m(\theta; a). \quad (19)$$

This is true since, for $\theta \in (0, \pi)$ and $a \in [0, 1)$ fixed, $f_m(\theta; a)$ is increasing in $m > 0$. Indeed, the differentiation yields

$$\frac{\partial f_m(\theta; a)}{\partial m} = \frac{a^2(1 - \cos \theta)}{[m(1 - a) + a]^2} > 0.$$

Now, we can prove the statement of the lemma. With the aid of (18), for $\theta \in (0, \pi)$ and $a \in [0, 1)$, we obtain

$$g_m(\theta; a) = \frac{|\sin(m\theta)[1 - a \cos(\theta)] + a \sin(\theta) \cos(m\theta)|}{\sin(\theta) \sqrt{1 + a^2 - 2a \cos(\theta)}} \leq \frac{m[1 - a \cos(\theta)] + a}{\sqrt{1 + a^2 - 2a \cos(\theta)}}.$$

It follows that the statement of the lemma holds provided that

$$\frac{m[1 - a \cos(\theta)] + a}{m(1 - a) + a} \leq \frac{\sqrt{1 + a^2 - 2a \cos(\theta)}}{1 - a},$$

which, in turn, is true if

$$1 - a \cos(\theta) \leq \sqrt{1 + a^2 - 2a \cos(\theta)}$$

by (19). By squaring, the last inequality is equivalent to $a^2 \cos^2(\theta) \leq a^2$, which is always true. \square

Now, we are in a position to prove the theorem on a spectral stability of the discrete Robin Schrödinger operator on \mathbb{N} . Recall that, by Theorem 1, $\sigma(J_a) = [-2, 2]$ for $a \in [0, 1)$.

Theorem 17. *Let $a \in [0, 1)$, $v = \{v_n\}_{n=1}^\infty$ be a complex sequence, and $V = \text{diag}(v_1, v_2, \dots)$. If the semi-infinite matrix K' with elements*

$$K'_{m,n} := \sqrt{|v_m|} \left[\frac{a}{1-a} + \min(m, n) \right] \sqrt{|v_n|}, \quad m, n \in \mathbb{N},$$

regarded as an operator on $\ell^2(\mathbb{N})$, satisfies $\|K'\| < 1$, then $\sigma(J_a + V) = \sigma_c(J_a + V) = \sigma(J_a)$. Equivalently, if there exists a constant $c < 1$ such that

$$|V| \leq c(-\Delta_a), \quad (20)$$

we have $\sigma(J_a + V) = \sigma_c(J_a + V) = \sigma(J_a)$.

Proof. Let $a \in [0, 1)$. For all $m, n \in \mathbb{N}$, we prove that

$$\sup_{z \in \mathbb{C}} |(J_a - z)_{m,n}^{-1}| = |(J_a - 2)_{m,n}^{-1}| = \frac{a}{1-a} + \min(m, n). \quad (21)$$

The verification of (21) is postponed to the end of the proof. Having (21), we may proceed similarly as in the proof of Theorem 3 and estimate the norm of the Birman–Schwinger operator this time as follows:

$$\begin{aligned} |\langle \phi, K(z)\psi \rangle| &\leq \sum_{m,n=1}^{\infty} |\phi_n| \sqrt{|v_m|} |(J_a - z)_{m,n}^{-1}| \sqrt{|v_n|} |\psi_n| \\ &\leq \sum_{m,n=1}^{\infty} |\phi_n| \sqrt{|v_m|} \left[\frac{a}{1-a} + \min(m, n) \right] \sqrt{|v_n|} |\psi_n| \\ &= \langle |\phi|, K' |\psi| \rangle \end{aligned}$$

for any $\phi, \psi \in \ell^2(\mathbb{N})$ and $z \in \mathbb{C}$. It follows that $\|K(z)\| \leq \|K'\|$ for all $z \in \mathbb{C}$. Hence, if $\|K'\| < 1$, the spectral stability follows by [15, Theorem 3].

Next, we prove the equivalence between the inequality $\|K'\| < 1$ and (20). The operator inequality (20) can be written as

$$\langle \psi, |V|\psi \rangle \leq c \langle \psi, (2 - J_a)\psi \rangle,$$

for all $\psi \in \ell^2(\mathbb{N})$, which holds true if and only if

$$\| |V|^{1/2} \psi \|^2 \leq c \| (2 - J_a)^{1/2} \psi \|^2,$$

for all $\psi \in \ell^2(\mathbb{N})$. Yet another equivalent form of the inequality reads

$$\| |V|^{1/2} (2 - J_a)^{-1/2} \phi \|^2 \leq c \| \phi \|^2,$$

for all $\phi \in \text{Ran}(2 - J_a)^{1/2}$. Since $2 \in \sigma_c(J_a)$, see Theorem 1, the range of $(2 - J_a)^{1/2}$ is dense in $\ell^2(\mathbb{N}_0)$. This means that $|V|^{1/2}(2 - J_a)^{-1/2}$ extends to a bounded operator with the norm

$$\left\| |V|^{1/2}(2 - J_a)^{-1/2} \right\| \leq \sqrt{c}.$$

Finally, it suffices to note that

$$\|K'\| = \left\| |V|^{1/2}(2 - J_a)^{-1}|V|^{1/2} \right\| = \left\| |V|^{1/2}(2 - J_a)^{-1/2} \right\|^2 \leq c < 1,$$

where we have used the fact that $\|TT^*\| = \|T\|^2$ for a bounded operator T .

It remains to verify (21). By inspection of formula (4), one observes that the Green kernel $(J_a - k - k^{-1})_{m,n}^{-1}$ is an analytic function in k in the open unit disk and continuous to its boundary. Indeed, the fact that the Green kernel extends continuously also to the points $k = \pm 1$ follows from (4) and limit relations

$$\frac{(k - a)k^{m+n} - (1 - ak)k^{n-m+1}}{(1 - ak)(1 - k^2)} \rightarrow \begin{cases} -\frac{m-a(m-1)}{1-a} & \text{as } k \rightarrow 1, \\ (-1)^{m+n} \frac{m+a(m-1)}{1+a} & \text{as } k \rightarrow -1, \end{cases}$$

that can be verified by a straightforward computation. Thus, by the Maximum Modulus Principle, one has

$$\sup_{z \in \mathbb{C}} |(J_a - z)_{m,n}^{-1}| = \max_{|k| \leq 1} |(J_a - k - k^{-1})_{m,n}^{-1}| = \max_{|k|=1} |(J_a - k - k^{-1})_{m,n}^{-1}|.$$

By setting $k = e^{i\theta}$, for $\theta \in [-\pi, \pi]$, and using (4), we get

$$\left| (J_a - 2 \cos \phi)_{m,n}^{-1} \right| = \left| \frac{(e^{i\theta} - a)e^{i(m+n-1)\theta} - (e^{-i\theta} - a)e^{i(-m+n+1)\theta}}{(1 - ae^{i\theta})(e^{i\theta} - e^{-i\theta})} \right| = g_m(\theta; a),$$

if $m \leq n$, where the function $g_m(\cdot; a)$ is defined in (17). By Lemma 16, we conclude that

$$\max_{|k|=1} |(J_a - k - k^{-1})_{m,n}^{-1}| = g_m(0; a) = \frac{a}{1-a} + m,$$

if $1 \leq m \leq n$. Equation (21) now follows from the symmetry of the Green kernel in m and n . \square

Remark 18. The stability of the spectrum of $J_a + V$ under the condition $\|K'\| < 1$ goes back to an old smoothness idea of Kato's [17]. In fact, it follows by his [17, Theorem 1.5] that $J_a + V$ is similar to J_a under the condition $\|K'\| < 1$. The stability of the spectrum of $J_a + V$ under the subordination condition (20) goes back to the idea of [12, Theorem 1] in a continuous setting. The equivalence between the two conditions was realized in [15], to where we refer for a survey, further abstract developments and applications.

Remark 19. The claim of Theorem 17 can be readily generalized to $a \in (-1, 1)$. In fact, the condition $\|K'\| < 1$ implies $\sigma(J_a + V) = \sigma_c(J_a + V) = \sigma(J_a)$ also for $a \in (-1, 0)$ because $\sigma(J_{-a} + V) = -\sigma(J_a - V)$ and similarly for each part of the spectra, which follows from Proposition 2, and the

operator K' is defined in terms of the absolute value $|V|$. On the other hand, condition (20) has to be replaced by $c(4 + \Delta_a) \geq |V|$ if $a \in (-1, 0)$, which is again a consequence of Proposition 2.

Corollary 20. *Suppose $a \in (-1, 1)$. If the potential $V = \text{diag}(v_1, v_2, \dots)$ satisfies*

$$\sum_{m,n=1}^{\infty} |v_m| \left[\frac{a}{1-a} + \min(m, n) \right]^2 |v_n| < 1, \quad (22)$$

or even the stricter condition

$$\sum_{n=1}^{\infty} \left(\frac{a}{1-a} + n \right) |v_n| < 1, \quad (23)$$

then $\sigma(J_a + V) = \sigma_c(J_a + V) = \sigma(J_a)$.

Proof. First, note that the left-hand side of (22) coincides with the square of the Hilbert–Schmidt norm of the operator K' from Theorem 17. Then (22) implies $\|K'\| \leq \|K'\|_{\text{HS}} < 1$ and hence the claim follows from Theorem 17 and Remark 19. Second, it suffices to note that (23) implies (22) which is a consequence of the inequality $(\alpha + \min(m, n))^2 \leq (\alpha + m)(\alpha + n)$ that holds true for all $m, n \in \mathbb{N}$ and $\alpha \geq 0$. \square

Note that the subordination of the sufficient conditions of Theorem 17 and Corollary 20 is as follows: (20) \Leftarrow (22) \Leftarrow (23).

Remark 21. It is interesting to compare Theorem 17 with its continuous analogue. Since the latter is not available in [13] (nor [11]), we establish the result here. Alternative conditions established by a completely different technique (including higher dimensions) can be found in [5].

Let H_α denote the Laplacian in $L^2((0, \infty))$, subject to the Robin boundary condition $\psi'(0) = \alpha\psi(0)$ with $\alpha \in \mathbb{C}$. By convention, the case $\alpha = +\infty$ is included as the Dirichlet Laplacian. More specifically, $\text{Dom}(H_\alpha) := \{\psi \in W^{2,2}((0, \infty)) \mid \psi'(0) = \alpha\psi(0)\}$ if $\alpha \in \mathbb{C}$ and $\text{Dom}(H_\alpha) := \{\psi \in W^{2,2}((0, \infty)) \mid \psi(0) = 0\}$ if $\alpha = +\infty$. For every $z \in \mathbb{C} \setminus [0, +\infty)$, the resolvent $(H_\alpha - z)^{-1}$ is the integral operator with kernel

$$G_z(x, x') := \frac{e^{-\sqrt{-z}|x-x'|} - e^{-\sqrt{-z}|x+x'|}}{2\sqrt{-z}} + \frac{e^{-\sqrt{-z}|x+x'|}}{\sqrt{-z} + \alpha},$$

where we consider the principal branch of the square root.

In analogy with Theorem 17, let us now restrict to real $\alpha \in (0, +\infty]$. Then the spectrum of H_α equals $[0, +\infty)$ and it is purely continuous. Let $V \in L^1_{\text{loc}}((0, \infty))$ be relatively form bounded with respect to H_α with the relative bound less than 1. Define $H_V := H_\alpha \dot{+} V$, where the sum on the right-hand side should be interpreted in the sense of forms.

It is not difficult to see that the pointwise inequality

$$|G_z(x, x')| \leq |G_0(x, x')| = \frac{1}{\alpha} + \frac{||x - x'| - |x + x'||}{2} = \frac{1}{\alpha} + \min(x, x')$$

holds true for every $z \in \mathbb{C} \setminus [0, +\infty)$ and $x, x' \in (0, \infty)$. Applying the Birman–Schwinger principle [15, Theorem 3], the spectral stability

$$\sigma(H_V) = \sigma_c(H_V) = \sigma(H_0)$$

holds true (in particular, the point spectrum is empty) whenever the integral operator K' with kernel

$$|V(x)|^{1/2} \left(\frac{1}{\alpha} + \min(x, x') \right) |V(x')|^{1/2}$$

has norm strictly less than one, or equivalently, there exists a constant $c < 1$ such that

$$\int_0^\infty V(x) |u(x)|^2 dx \leq c \left(\int_0^\infty |u'(x)|^2 dx + \alpha |u(0)|^2 \right) \quad (24)$$

for every $u \in W^{1,2}((0, \infty))$ if $\alpha > 0$ or $u \in W_0^{1,2}((0, \infty))$ if $\alpha = +\infty$ (the term $\alpha |u(0)|^2$ is interpreted as zero in the latter case). Note that this subordination condition particularly ensures that V is relatively form bounded with respect to H_α with the relative bound less than one. Estimating the operator norm of K' by the Hilbert–Schmidt norm, a sufficient condition reads

$$\iint_{(0, \infty)^2} |V(x)| \left(\frac{1}{\alpha} + \min(x, x') \right)^2 |V(x')| dx dx' < 1. \quad (25)$$

Since $\min(x, x')^2 \leq xx'$, for $x, x' \geq 0$, an obvious sufficient condition to guarantee (25) reads

$$\int_0^\infty |V(x)| \left(\frac{1}{\alpha} + x \right) dx < 1. \quad (26)$$

Obviously, the sufficient conditions (24)–(26) are continuous analogues of (20), (22) and (23), respectively.

Although the weaker assumption $\|K'\| \leq 1$, where K' is as in Theorem 17, does not guarantee $\sigma_p(J_a + V) = \emptyset$, for $a \in [0, 1)$, it follows at least that the discrete spectrum $\sigma_d(J_a + V)$ is empty. In other words, we still have $\sigma(J_a + V) = [-2, 2]$ but the spectrum of $J_a + V$ need not be purely continuous, that is, the existence of eigenvalues embedded in the interval $[-2, 2]$ cannot be excluded.

Theorem 22. Suppose that $a \in [0, 1)$ and $\{v_n\}_{n=1}^\infty \subset \mathbb{C}$ is such that $\|K'\| \leq 1$, where K' is as in Theorem 17, then $\sigma_d(J_a + V) = \emptyset$. Equivalently, if

$$|V| \leq -\Delta_a, \quad (27)$$

then $\sigma_d(J_a + V) = \emptyset$.

Proof. Suppose $V = \text{diag}(v_1, v_2, \dots)$ is such that $\|K'\| \leq 1$. For $q \in (0, 1)$, we define an auxiliary operator K'_q corresponding to the potential qV , that is, $K'_q = qK'$. Since

$$\|K'_q\| = q\|K'\| \leq q < 1,$$

we have $\sigma_d(J_a + qV) = \emptyset$ for all $q \in (0, 1)$ by Theorem 17.

Clearly, $J_a + qV \rightarrow J_a + V$ uniformly as $q \rightarrow 1^-$. Consequently, if there exists $\lambda \in \sigma_d(J_a + V)$, then there must be a discrete eigenvalue of $J_a + qV$ in a neighborhood of λ for q sufficiently close to 1 contradicting $\sigma_d(J_a + qV) = \emptyset$. Therefore, $\sigma_d(J_a + V) = \emptyset$.

The proof of the equivalence between the condition $\|K'\| \leq 1$ and (27) follows the same lines as in the proof of Theorem 17. \square

Remark 23. Analogically as in Remark 19, the first statement of Theorem 22 holds true with no change even for $a \in (-1, 0)$, while condition (27) is to be replaced by the inequality $4 + \Delta_a \geq |V|$ if $a \in (-1, 0)$.

The following statement can be deduced from Theorem 22 similarly as Corollary 20 from Theorem 17.

Corollary 24. *If $a \in (-1, 1)$ and potential $V = \text{diag}(v_1, v_2, \dots)$ fulfills*

$$\sum_{m,n=1}^{\infty} |v_m| \left[\frac{a}{1-a} + \min(m, n) \right]^2 |v_n| \leq 1, \quad (28)$$

or even

$$\sum_{n=1}^{\infty} \left(\frac{a}{1-a} + n \right) |v_n| \leq 1,$$

then $\sigma_d(J_a + V) = \emptyset$.

Finally, we may deduce more concrete conditions on the potential V guaranteeing the spectral stability of $J_a + V$ by combining Theorems 17 and 22 with the Hardy weights given in Theorem 15.

Theorem 25. *Let $a \in [0, 1)$. If the complex potential $V = \text{diag}(v_1, v_2, \dots)$ satisfies*

$$|v_n| \leq c \left[2 - \left(1 - \frac{1}{n} \right)^q - \left(1 + \frac{1}{n} \right)^q - a\delta_{n,1} \right], \quad \forall n \in \mathbb{N}, \quad (29)$$

for a constant $c < 1$ and $q \in (0, q_a]$, where q_a is defined by (16), then

$$\sigma(J_a + V) = \sigma_c(J_a + V) = [-2, 2].$$

If condition (29) is fulfilled with $c = 1$, then $\sigma_d(J_a + V) = \emptyset$.

Remark 26. Sufficient conditions (22) and (29) are not comparable and similarly for (28) and (29) with $c = 1$. Let us consider the latter case and define two potential sequences as follows:

$$v_n := \begin{cases} 0, & \text{if } n = 1, \\ \frac{q(1-q)}{n^2}, & \text{if } n \geq 2, \end{cases} \quad \text{and} \quad \tilde{v}_n := \left(\frac{a}{1-a} + n_0 \right)^{-1} \delta_{n_0, n}, \quad n \in \mathbb{N},$$

where $n_0 \geq 2$ is large enough to fulfill

$$2 - \left(1 - \frac{1}{n_0} \right)^q - \left(1 + \frac{1}{n_0} \right)^q < \left(\frac{a}{1-a} + n_0 \right)^{-1},$$

which is possible since the left-hand side decays as $1/n_0^2$. Then v satisfies (29) with $c = 1$ but the series in (28) diverges. On the other hand, \tilde{v} satisfies (28), even the stricter condition of Corollary 24, but not (29).

6 | AN OPEN PROBLEM

As a final remark, we emphasize an interesting research problem. It is related to the possibility of the extension of the present method for discrete Schrödinger operators on the half-line which are made critical by subtracting the optimal Hardy weight.

In the continuous setting, the Schrödinger operator with the optimal Hardy potential

$$H := -\frac{d^2}{dx^2} - \frac{1}{4x^2}$$

acting in $L^2((0, \infty))$ and subject to Dirichlet boundary condition at $x = 0$ is critical, which means that by adding an arbitrary negative potential to H destroys the positivity of the operator. Since the eigenvalue problem for H turns out to be related to a particular Bessel differential equation, spectral properties of H can be deduced in terms of well-known special functions, see, for example, [6]. Then one may consider H as an unperturbed operator and investigate, for example, spectral enclosures for perturbations of H by small complex potentials, asymptotic analysis of discrete eigenvalues of the perturbed operator under various settings, etc.

A discrete variant of the continuous problem above regards to spectral properties of the Jacobi operator $J_0 - W$ with the perturbing potential determined by the optimal discrete Hardy weight (12). However, solutions of the difference equation of the eigenvalue problem for $J_0 - W$ do not seem to be expressible in terms of known special functions. Consequently, no explicit formula for the Green kernel of $J_0 - W$ seems to be available. A more detailed spectral analysis of the critical operator $J_0 - W$ would be of great interest. In addition to the perturbation analysis mentioned above, another motivation comes from theory of orthogonal polynomials. Indeed, the corresponding family of orthogonal polynomials determined by the recurrence

$$p_{n+1}(x) = \left(x - \sqrt{1 - \frac{1}{n}} - \sqrt{1 + \frac{1}{n}} \right) p_n(x) - p_{n-1}(x), \quad n \in \mathbb{N},$$

with $p_0(x) = 1$ and $p_1(x) = x - \sqrt{2}$, seems not to have been analyzed yet.

APPENDIX: ILLUSTRATIVE AND COMPARISON PLOTS

Below in Figures A.1–A.3, we provide illustrative plots of the optimal spectral enclosures of Theorem 3 for $a \in \{1/2, 2, i(1 + \sqrt{5})/2\}$. Namely, the plots show the boundary curves given by the equation

$$\sqrt{|z^2 - 4|} = g_a(z)Q,$$

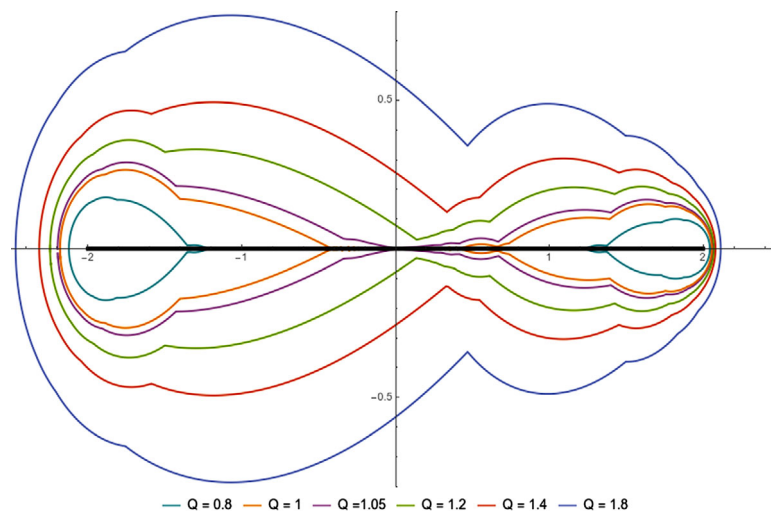


FIGURE A.1 Optimal spectral enclosures (7) for $a = 1/2$ and several values of $Q = \|v\|_{\ell^1}$

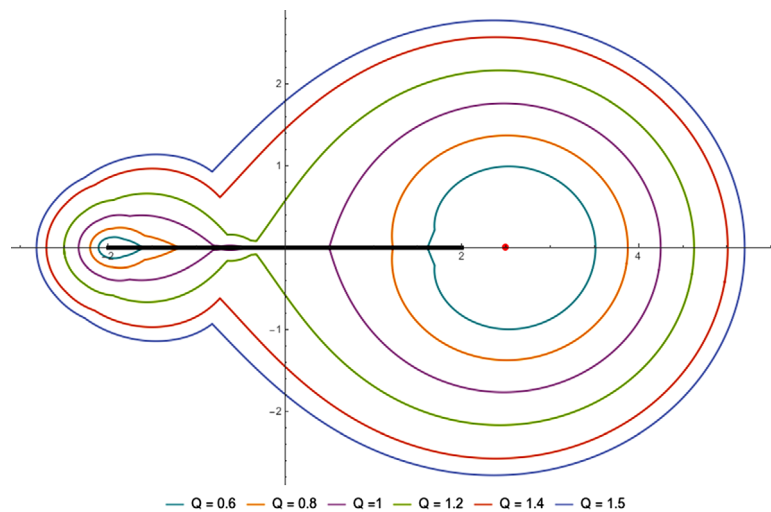


FIGURE A.2 Optimal spectral enclosures (7) for $a = 2$ and several values of $Q = \|v\|_{\ell^1}$. The red dot designates the sole eigenvalue of J_a

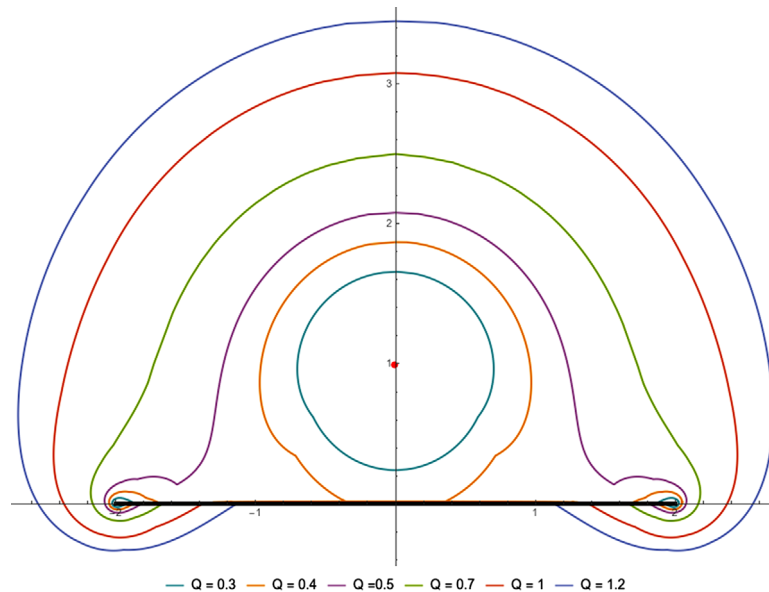


FIGURE A.3 Optimal spectral enclosures (7) for $a = i(1 + \sqrt{5})/2$ and several values of $Q = \|v\|_{\ell^1}$. The red dot designates the sole eigenvalue of J_a

for several values of the parameter $Q = \|v\|_{\ell^1}$. For $a = 2$ and $a = i(1 + \sqrt{5})/2$, the spectrum of the unperturbed operator J_a contains the extra eigenvalue $a + a^{-1}$, see Theorem 1, that is designated by a red dot.

It would be interesting to know more about geometric and topological properties of the optimal spectral enclosures of Theorem 3. A serious analysis could be a subject of future research.

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