ON A CONJECTURE BY HUNDERTMARK AND SIMON

ARI LAPTEV, MICHAEL LOSS, AND LUKAS SCHIMMER

Abstract. The main result of this paper is a complete proof of a new Lieb-Thirring type inequality for Jacobi matrices originally conjectured by Hundertmark and Simon. In particular it is proved that the estimate on the sum of eigenvalues does not depend on the off-diagonal terms as long as they are smaller than their asymptotic value. An interesting feature of the proof is that it employs a technique originally used by Hundertmark-Laptev-Weidl concerning sums of singular values for compact operators. This technique seems to be novel in the context of Jacobi matrices.

1. Introduction

In this note we prove a conjecture of Hundertmark and Simon [HS] concerning a sharp Lieb-Thirring inequality for Jacobi matrices. We denote the symmetric Jacobi matrix with diagonal entries $\{b_n\}_{n=-\infty}^{\infty}$ and off-diagonal entries $\{a_n\}_{n=-\infty}^{\infty}$ by

$$J := W(\{a_n\}, \{b_n\}) \ .$$

It is assumed that the a_n tend to 1 as $n \to \pm\infty$ which yields the interval $[-2, 2]$ as the essential spectrum of this Jacobi matrix. We denote by $E_j^+(J)$ the eigenvalues of J that are larger than 2 and by $E_j^-(J)$ the eigenvalues of J that are less than -2. Hundertmark and Simon proved that

$$\sum_j (E_j^+(J)^2 - 4)^{1/2} + (E_j^-(J)^2 - 4)^{1/2} \le \sum_n |b_n| + 4 \sum_n |a_n - 1| \ , \quad (1.1)$$

and observed that this inequality is sharp. Indeed, in the absence of the potential, they noted that the Jacobi matrix with all entries $a_n = 1$ except for a single one that is chosen to be larger than one, yields equality in (1.1). They then conjectured the improved version of (1.1) in which $|a_n - 1|$ is replaced by $(a_n - 1)_+$, where we use the notation $(a)_+$ to mean a if $a > 0$ and 0 if $a \le 0$. We have the following theorem.

2010 *Mathematics Subject Classification.* 35P15, 47B36, 47A10.
Key words and phrases. Lieb–Thirring inequalities, Jacobi matrices.

THEOREM 1.1. Assume that $\sum_n |b_n| < \infty$, $\sum_n (a_n - 1)_+ < \infty$ and $\lim_{n\to\pm\infty} a_n = 1$. Then we have the bound on the eigenvalue sum

$$\sum_j (E_j^+(J)^2 - 4)^{1/2} + (E_j^-(J)^2 - 4)^{1/2} \le \sum_n |b_n| + 4\sum_n (a_n - 1)_+ \ . \quad (1.2)$$

The following consequence provides another justification for this short note. Generally, the proof of Lieb-Thirring inequalities is patterned after the ones for the continuum in which case the kinetic energy is given by $-\Delta$. The discrete Laplacian requires that all $a_n = 1$. It is, however, of some interest that in the case of Jacobi matrices this needs not be the case. This is a distinguishing feature of Lieb-Thirring inequalities for Jacobi matrices. Further, it was shown in [KS], (see also [S]) that

$$\sum_j F(E_j^+(J)) + F(E_j^-(J)) \le \sum_n b_n^2 + 2G(a_n)^2, \quad (1.3)$$

where $G(a) = a^2 - 1 - \log|a|^2$ and $F(E) = \beta^2 - \beta^{-2} - \log|\beta|^2$ and $E = \beta + \beta^{-1}$ with $|\beta| > 1$.

In [Sch] one of us proved that if $a_n \equiv 1$ then the inequality (1.1) implies (1.3). Using this argument and Theorem 1.1 we obtain as a consequence

THEOREM 1.2. Let $\gamma > 1/2$. Assume that $\sum_n b_n^{\gamma+1/2} < \infty$ and that $a_n \ge 0$ for all $n \in \mathbb{Z}$ with $\lim_{n\to\pm\infty} a_n = 1$ and $\sum_n (a_n - 1)_+^{\gamma+1/2} < \infty$. Then with $\mathrm{B}(x,y)$ denboting the Beta function

$$\sum_j \int_2^{|E_j^\pm(J)|} (t^2-4)^{\frac12} (|E_j^\pm(J)| - t)^{\gamma-\frac32}\, dt$$
$$\le \mathrm{B}(\gamma - 1/2, 2) \sum_n (\pm[b_n]_\pm \pm [a_n - 1]_+ \pm [a_{n-1} - 1]_+)_\pm^{\gamma+\frac12} \ . \quad (1.4)$$

For $\gamma = 3/2$ the left-hand side coincides with $\frac12 \sum_j F(E_j^\pm(J))$. The function $G(a) \ge 0$ equals zero if and only if $a = \pm 1$ and hence (1.4) is an improvement over (1.3) for the case where $0 \le a_n \le 1$.

Remark. As proved in [Sch], the left-hand side in (1.4) is bounded from below by

$$\sum_j \int_2^{|E_j^\pm|} (t^2-4)^{\frac12} (|E_j^\pm| - t)^{\gamma-\frac32}\, dt \ge 2\mathrm{B}(\gamma - 1/2, 3/2) \sum_j (|E_j^\pm| - 2)^\gamma$$

and by

$$\sum_j \int_2^{|E_j^\pm|} (t^2-4)^{\frac{1}{2}} (|E_j^\pm| - t)^{\gamma - \frac{3}{2}} \, dt \geq \mathrm{B}(\gamma - 1/2, 2) \sum_j (|E_j^\pm| - 2)^{\gamma + \frac{1}{2}} \, .$$

Thus (1.4) improves on corresponding Lieb-Thirring inequalities in [HS]. Note that in [HS, p.121] an argument is given that allows to replace $(a_n - 1)$ in their results by $(a_n - 1)_+$ for $\gamma \geq 1$ but importantly not for $1/2 < \gamma < 1$ and not in the case of the main result (1.2).

In order to prove Theorem 1.1 we reduce the problem (as in [HS]) to the discrete Schrödinger operator. When treating terms $0 \leq a_n \leq 1$ we use some additional convexity property.

2. The proof of the main result.

We follow Hundertmark and Simon except for one key step. Using norm resolvent convergence we may assume that only finitely many of the a_n are not equal to one and finitely many b_n are not equal to zero. Likewise, we may assume that $b_n \geq 0$. We also write

$$W(\{a_n\}, \{b_n\}) = A + B$$

with the understanding that A contains only the off-diagonal terms and B the diagonal terms of the Jacobi matrix. If the off-diagonal terms are all equal to one we denote the corresponding matrix by A_1. The essential spectrum is given by the interval $[-2, 2]$ which follows from Weyl's theorem. Let us denote the eigenvalues that are strictly greater than 2 by $E_1^+(A+B) \geq E_2^+(A+B) \geq E_3^+(A+B) \geq \cdots$. In what follows, the eigenvalues that are strictly less than -2 can be treated in a similar fashion.

Treating $a_n > 1$*:* Consider the window of the matrix A that contains an off-diagonal term $a > 1$ and use the elementary inequalities

$$\begin{pmatrix} -a+1 & 1 \\ 1 & -a+1 \end{pmatrix} \leq \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \leq \begin{pmatrix} a-1 & 1 \\ 1 & a-1 \end{pmatrix}.$$

Applying it to all $a_n > 1$ we obtain

$$\widetilde{J}^- = W(\{\widetilde{a}_n\}, \{\widetilde{b}_n^-\} \leq J = W(\{a_n\}, \{b_n\}) \leq W(\{\widetilde{a}_n\}, \{\widetilde{b}_n^+\} = \widetilde{J}^+,$$

where

$$\widetilde{a}_n = \begin{cases} a_n, & \text{if } a_n \leq 1 \\ 1, & \text{if } a_n > 1 \end{cases} \qquad \widetilde{b}_n^\pm = \pm[b_n]_\pm \pm [(a_{n-1} - 1)_+ + (a_n - 1)_+] \text{ for. all } n \, ,$$

where $[x]_\pm = \max\{\pm x, 0\}$. This implies

$$\sum_j (E_j^+(J)^2 - 4)^{1/2} \le \sum_j (E_j^+(\widetilde{J}^+)^2 - 4)^{1/2} \tag{2.1}$$

and similarly

$$\sum_j (E_j^-(J)^2 - 4)^{1/2} \le \sum_j (E_j^-(\widetilde{J}^-)^2 - 4)^{1/2} \ .$$

This reduces the problem to the case $a_n \le 1$, $n \in \mathbb{Z}$.

Treating $a_n \le 1$*:* Assuming $a_n \le 1$ we consider the Birman-Schwinger operator

$$K(A;\beta) := B^{1/2}(\beta - A)^{-1}B^{1/2},$$

where $\beta > 2$ and list the eigenvalues of the Birman-Schwinger operator, $E_j(B^{1/2}(\beta - A)^{-1}B^{1/2})$, in decreasing order. The Birman-Schwinger principle states that the jth eigenvalue of $B^{1/2}(E_j^+(A+B) - A)^{-1}B^{1/2}$ is one.

Let us decompose the matrix A in a certain fashion. Consider the following window of the general matrix A_κ,

$$\begin{pmatrix} 0 & a & 0 & 0 & 0 \\ a & 0 & \kappa & 0 & 0 \\ 0 & \kappa & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ 0 & 0 & 0 & d & 0 \end{pmatrix} .$$

The distinct notation κ indicates that we concentrate on this particular position of the matrix. Denote by U the infinite diagonal matrix that consists of $+1$ on the diagonal above the position of κ and of -1 below κ, i.e., its window is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} .$$

It has the effect that the corresponding window of the matrix $U\,A_\kappa\,U$ is given by

$$\begin{pmatrix} 0 & a & 0 & 0 & 0 \\ a & 0 & -\kappa & 0 & 0 \\ 0 & -\kappa & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ 0 & 0 & 0 & d & 0 \end{pmatrix} ,$$

i.e., the entry κ changes sign and all others are unchanged. Since U is unitary, the matrices A and UAU are unitarily equivalent and have the same spectrum. With a slight abuse of notation we now identify the matrices with their window. If we assume that $0 \le \kappa < 1$, we may write

$$A_\kappa = \frac{\kappa+1}{2}\begin{pmatrix} 0 & a & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ 0 & 0 & 0 & d & 0 \end{pmatrix} + \frac{1-\kappa}{2}\begin{pmatrix} 0 & a & 0 & 0 & 0 \\ a & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & c & 0 \\ 0 & 0 & c & 0 & d \\ 0 & 0 & 0 & d & 0 \end{pmatrix},$$

or if we denote by A' the first matrix displayed above we can write

$$A_\kappa = \frac{1+\kappa}{2}A' + \frac{1-\kappa}{2}UA'U\ .$$

Repeating this for all the off-diagonal elements that are strictly less than one we find

$$A = \sum_j \lambda_j U(j)\, A_1\, U(j), \tag{2.2}$$

where off-diagonal elements of A_1 are equal one and where $\lambda_j \ge 0$, $\sum_j \lambda_j = 1$. Since the matrices $U(j)$ are diagonal and have the matrix elements ± 1, the matrices A_1 and $U(j)A_1U(j)$ have the same eigenvalues.

The key observation is the following lemma

LEMMA 2.1. Let $\beta I > X$. Then the function

$$X \to (\beta I - X)^{-1}$$

is operator convex, i.e., if $0 \le \lambda \le 1$ then

$$(\beta - \lambda X_1 - (1-\lambda)X_2)^{-1} \le \lambda(\beta - X_1)^{-1} + (1-\lambda)(\beta - X_2)^{-1}\ .$$

Proof. We follow [HP]. Let $Y_j = \beta - X_j$, $j = 1, 2$. It amounts to showing that for two positive and invertible self-adjoint operators Y_1 and Y_2 we have

$$(\lambda Y_1 + (1-\lambda)Y_2)^{-1} \le \lambda Y_1^{-1} + (1-\lambda)Y_2^{-1}\ .$$

This is equivalent to

$$\left[Y_2^{1/2}\left(\lambda Y_2^{-1/2}Y_1Y_2^{-1/2} + (1-\lambda)I\right)Y_2^{1/2}\right]^{-1} \le Y_2^{-1/2}\left[\lambda Y_2^{1/2}Y_1^{-1}Y_2^{1/2} + (1-\lambda)I\right]Y_2^{-1/2}$$

or

$$Y_2^{-1/2}\left(\lambda Y_2^{-1/2}Y_1Y_2^{-1/2}+(1-\lambda)I\right)^{-1}Y_2^{-1/2} \\ \leq Y_2^{-1/2}\left[\lambda Y_2^{1/2}Y_1^{-1}Y_2^{1/2}+(1-\lambda)I\right]Y_2^{-1/2}$$

which is equivalent to

$$\left(\lambda Y_2^{-1/2}Y_1Y_2^{-1/2}+(1-\lambda)I\right)^{-1}\leq \lambda Y_2^{1/2}Y_1^{-1}Y_2^{1/2}+(1-\lambda)I\ .$$

This is an inequality in terms of the positive, invertible and self-adjoint operator $Y=Y_2^{-1/2}Y_1Y_2^{-1/2}$, i.e.,

$$(\lambda Y+(1-\lambda)I)^{-1}\leq \lambda Y^{-1}+(1-\lambda)I,$$

which reduces the whole problem to positive numbers on account of the spectral theorem. For positive numbers the inequality is obvious. □

Applying now Lemma 2.1 to (2.2) we find

$$K(A;\beta)\leq \sum_j \lambda_j U(j)K(A_1;\beta)U(j)\ .$$

If we set $\beta=\mu+\frac{1}{\mu}$ and introduce the operator

$$L_\mu(A):=(\beta^2-4)^{1/2}K(A;\beta)$$

we find

$$L_\mu(A)\leq \sum_j \lambda_j U(j)L_\mu(A_1)U(j)\ .$$

The operator $L_\mu(A_1)$ has the matrix representation

$$[L_\mu(A_1)]_{m,n}=b_m^{1/2}\mu^{|n-m|}b_n^{1/2}$$

where, once more

$$\beta=\mu+\frac{1}{\mu}\ ,\mu<1\ .$$

Denote by $S_n(\mu)$ the sum of the n largest eigenvalues of $\sum_j \lambda_j U(j)L_\mu(A_1)U(j)$.

LEMMA 2.2.

$$S_n(\mu)\leq S_n(\nu)$$

for $\nu\geq\mu$.

We will present two proofs of this Lemma.

Proof of Lemma 2.2 following Hundertmark and Simon [HS]. Pick any bounded sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ and consider the matrix

$$M_{\{\mu_n\}} := \sum_j \lambda_j U(j) L_{\{\mu_n\}}(A_1) U(j) \ ,$$

where

$$(L_{\{\mu_n\}})_{k,\ell} = \begin{cases} b_k^{1/2} \mu_k \cdots \mu_{\ell-1} b_\ell^{1/2} & \text{if } k \le \ell \\ (L_{\{\mu_n\}})_{\ell,k} & \text{if } k > \ell \ . \end{cases}$$

Now we fix some integer n and set $\mu_n = \mu$. All the other ones are fixed. The matrix $\sum_j \lambda_j U(j) L_{\{\mu_n\}} U(j)$ is an affine function of μ with a diagonal that is independent of μ and hence the matrix is of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} + \mu \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix} \ .$$

Now we consider the sum of the top n eigenvalues of this matrix and denote this function by $f(\mu)$. This function is convex. Moreover, if we consider the diagonal matrix $V : \ell^2 \to \ell^2$ given by

$$[V\phi]_n = (-1)^n \phi_n$$

we find that

$$\begin{aligned} V \sum_j \lambda_j U(j) L_{\{\mu_n\}}(A_1) U(j) V &= \sum_j \lambda_j V U(j) L_{\{\mu_n\}}(A_1) U(j) V \\ &= \sum_j \lambda_j U(j) V L_{\{\mu_n\}}(A_1) V U(j) \end{aligned}$$

since the matrix V is also diagonal and commutes with the $U(j)$. This matrix has the same spectrum but is of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} - \mu \begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}$$

and hence $f(\mu)$ is even. Thus f, being convex, is monotone for $\mu > 0$. □

Remark. The above proof is closely related to the proof of a similar result from [HLTh] except for using some symmetry property rather than perturbation at the spectral point zero.

Proof of Lemma 2.2 following Hundertmark, Laptev and Weidl [HLW]. We aim to use the following abstract result of [HLW] concerning the sum

$\|T\|_n$ of the largest n singular values of a compact operator T,

$$\|T\|_n = \sum_{j=1}^{n} \sqrt{E_j(T^*T)}\,.$$

The result is an immediate consequence of $\|T\|_n$ defining a norm by Ky-Fan's inequality.

LEMMA 2.3. Let T be a non-negative compact operator on a Hilbert space $\mathcal{G}$, let g be a probability measure on Ω, and let $\{V(k)\}_{k\in\Omega}$ be a family of unitary operators on $\mathcal{G}$. Then, for any $n \in \mathbb{N}$,

$$\left\| \int_\Omega V(k)^* T V(k)\, dg(k) \right\|_n \le \|T\|_n\,.$$

To apply the above result to $L_\mu(A_1)$, we recall the unitary map $\mathcal{F}$: $L^2([-\pi,\pi]) \to \ell^2(\mathbb{Z})$ onto the Fourier coefficients

$$(\mathcal{F}u)_n = \widehat{u}_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}nk} u(k) dk\,, \qquad (\mathcal{F}^*u)(k) = \frac{1}{\sqrt{2\pi}} \sum_n u_n \mathrm{e}^{-\mathrm{i}nk}\,.$$

By means of the transform $\mathcal{F}$, the free operator W with $a_n \equiv 1, b_n \equiv 0$ is unitarily equivalent to the operator $-2\cos k$ on $L^2([-\pi,\pi])$. Defining g_μ to be the non-negative function

$$g_\mu(k) = \frac{1}{\sqrt{2\pi}} \frac{\frac{1}{\mu} - \mu}{-2\cos k + \frac{1}{\mu} + \mu}$$

and denoting by T the projection onto $b_n^{1/2}$, and by $V(k)$ the unitary operator $(V(k)u)_n = \mathrm{e}^{-\mathrm{i}nk} u_n$, we can thus write

$$L_\mu(A_1) = \int_{-\pi}^{\pi} V(k)^* T V(k) \frac{g_\mu(k)}{\sqrt{2\pi}} dk\,.$$

To obtain some more properties of g_μ we note that its Fourier transform is given by $(\widehat{g}_\mu)_n = \mu^{|n|}$. This can in particular be used to establish the aforementioned matrix representation of $L_\mu(A_1)$. For our purposes, we note that for $0 < \mu < \nu < 1$ clearly

$$(\widehat{g}_\mu)_0 = 1\,, \qquad \widehat{g}_\nu \widehat{g}_{\mu/\nu} = \widehat{g}_\mu\,. \tag{2.3}$$

Since g_μ is smooth and periodic in k, the pointwise identity

$$g_\mu(k) = \frac{1}{\sqrt{2\pi}} \sum_n (\widehat{g}_\mu)_n \mathrm{e}^{-\mathrm{i}nk}$$

holds and the properties (2.3) imply

$$\int_{-\pi}^{\pi} \frac{g_\mu(k)}{\sqrt{2\pi}}\, dk = 1\,, \qquad \frac{g_\nu}{\sqrt{2\pi}} * \frac{g_{\mu/\nu}}{\sqrt{2\pi}} = \frac{g_\mu}{\sqrt{2\pi}}$$

for all $0 < \mu < \nu < 1$. The convolution identity is understood in the sense that

$$\int_{-\pi}^{\pi} \frac{g_\nu(k-k')}{\sqrt{2\pi}} \frac{g_{\mu/\nu}(k')}{\sqrt{2\pi}}\, dk' = \int_{-\pi}^{\pi} \frac{g_\nu(k')}{\sqrt{2\pi}} \frac{g_{\mu/\nu}(k-k')}{\sqrt{2\pi}}\, dk' = \frac{g_\mu(k)}{\sqrt{2\pi}}$$

which is well-defined since all three functions are periodic. Using this identity together with the fact that $V(k'+k'') = V(k')V(k'')$ and that $V(k')$ and $U(j)$ commute as both are multiplication operators, we obtain

$$\begin{aligned} &\sum_j \lambda_j U(j) L_\mu(A_1) U(j) \\ &\quad = \sum_j \lambda_j \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} U(j) V(k)^* T V(k) U(j) \frac{g_\nu(k-k')}{\sqrt{2\pi}} \frac{g_{\mu/\nu}(k')}{\sqrt{2\pi}}\, dk'\, dk \\ &\quad = \int_{-\pi}^{\pi} V(k')^* \left(\sum_j \lambda_j U(j) \int_{-\pi-k''}^{\pi-k''} V(k'')^* T V(k'') \frac{g_\nu(k'')}{\sqrt{2\pi}}\, dk'' U(j) \right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times V(k') \frac{g_{\mu/\nu}(k')}{\sqrt{2\pi}}\, dk'\,. \end{aligned}$$

By periodicity, the operator

$$\int_{-\pi-k'}^{\pi-k'} V(k'')^* T V(k'') \frac{g_\nu(k'')}{\sqrt{2\pi}}\, dk'' = \int_{-\pi}^{\pi} V(k'')^* T V(k'') \frac{g_\nu(k'')}{\sqrt{2\pi}}\, dk'' = L_\nu(A_1)$$

is independent of k' and thus we can apply Lemma 2.3 to obtain the desired monotonicity. The special case $\nu = 1$ is an immediate consequence of Lemma 2.3. □

Remark. The above proof is closely related to the continuous case [HLW] with the established convolution identity for g_μ replacing the fact that the Cauchy distribution is a convolution semigroup. Using the parametrisation $\mu = \mathrm{e}^{-\sigma}, \nu = \mathrm{e}^{-\tau}$ with $\sigma > \tau > 0$ the above identity may also be written in the more similar form $\frac{g_\tau}{\sqrt{2\pi}} * \frac{g_{\sigma-\tau}}{\sqrt{2\pi}} = \frac{g_\sigma}{\sqrt{2\pi}}$.

Because of the Birman-Schwinger principle, $E_j(K(A;E_j^+))$, the j-the eigenvalue of $K(A;E_j^+)$, equals 1 and hence

$$\sum_{j=1}^n (E_j^{+2}(J)-4)^{1/2} = \sum_{j=1}^n (E_j^{+2}(J)-4)^{1/2} E_j(K(A;E_j^+)) = \sum_{j=1}^n E_j(L_{\mu_j}(A))$$

$$\leq \sum_{j=1}^n E_j(\sum_k \lambda_k U(k) L_{\mu_j}(A_1) U(k)) \leq S_n(\mu_n)$$

where $\mu_j + \frac{1}{\mu_j} = E_j^+$. Now, again, we proceed as in Hundertmark - Simon [HS] and get the estimate

$$S_{N^+}(\mu_n) \leq \mathrm{Tr} \sum_j \lambda_j U(j) L_{\mu=1}(A_1) U(j) = \mathrm{Tr} B \ ,$$

where $S_{N^+}(\mu_n)$ includes all the eigenvalues of $\widetilde{J}$ that are greater than 2. In other words

$$\sum_j (E_j^{+2}(\widetilde{J}^+) - 4)^{1/2} \leq \sum_n [b_n]_+ + (a_{n-1}-1)_+ + (a_n-1)_+ = \sum_n [b_n]_+ + 2(a_n-1)_+ \ .$$

As shown in [HS] the Jacobi matrices

$$W(\{a_n\},\{b_n\}) \text{ and } -W(\{a_n\},\{-b_n\})$$

are unitarily equivalent and hence it follows that

$$\sum_j (E_j^{-2}(\widetilde{J}^-) - 4)^{1/2} \leq \sum_n [b_n]_- + (a_{n-1}-1)_+ + (a_n-1)_+ = \sum_n [b_n]_- + 2(a_n-1)_+ \ ,$$

which together with the previous estimate proves Theorem 1.1.

As a corollary we obtain Theorem 1.2.

Proof of Theorem 1.2. Let $\chi_{\{c,d\}}$ be the characteristic function of the interval (c,d). Then, recalling $J \leq \widetilde{J}^+$, we have

$$\sum_j \int_2^{E_j^+} (t^2-4)^{\frac{1}{2}} (E_j^+ - t)^{\gamma - \frac{3}{2}} \, dt$$

$$= \sum_j \int_0^\infty \left((E_j^+(J-s))^2 - 4\right)^{\frac{1}{2}} s^{\gamma-\frac{3}{2}} \chi_{\{E_j^+(J-s)\geq 2\}}(s) \, ds$$

$$\leq \sum_j \int_0^\infty \left((E_j^+(\widetilde{J}^+-s))^2 - 4\right)^{\frac{1}{2}} s^{\gamma-\frac{3}{2}} \chi_{\{E_j^+(\widetilde{J}^+-s)\geq 2\}}(s) \, ds \, .$$

Applying first the variational principle and then the main result we obtain

$$\sum_j \int_2^{E_j^+} (t^2-4)^{\frac12} (E_j^+ - t)^{\gamma-\frac32}\, dt \\ \le \sum_n \int_0^\infty (\widetilde{b}_n^+ - s)_+ s^{\gamma-\frac32}\, ds = \mathrm{B}(\gamma-1/2,2) \sum_n (\widetilde{b}_n^+)_+^{\gamma+\frac12}$$

and the proof is complete.

□

Acknowledgment: Work of Michael Loss was partially supported by U.S. National Science Foundation grant DMS 1856645; Ari Laptev was partially supported by RSF grant 18-11-0032. Lukas Schimmer was supported by Vetenskapsrådet, grant 2017-04736.
Michael Loss would like to thank Institute Mittag-Leffler for its generous hospitality.

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(A. Laptev) Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK, and St Petersburg University14-Ya Liniya B.O., 29B St Petersburg 199178, Russia
E-mail address: a.laptev@imperial.ac.uk

(M. Loss) School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332 USA
E-mail address: loss@math.gatech.edu

(L. Schimmer) Institut Mittag–Leffler, The Royal Swedish Academy of Sciences, 182 60 Djursholm, Sweden
E-mail address: lukas.schimmer@kva.se