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ABSOLUTELY CONTINUOUS SPECTRUM OF A TYPICAL SCHRÖDINGER OPERATOR WITH AN OPERATOR-VALUED POTENTIAL

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In memory of Sergei Naboko, our friend and colleague

Abstract. The content of the paper is reflected by its title.

§1. Main results

Let $\mathfrak{H}$ be a separable Hilbert space and let V be a measurable function from $\mathbb{R}_+$ to the set of bounded selfadjoint operators on $\mathfrak{H}$. The measurability of V means that the function $x \mapsto \langle V(x)h, h\rangle$ is measurable for each $h \in \mathfrak{H}$. We study the absolutely continuous spectrum of the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + \alpha V, \quad V^* = V, \tag{1}$$

acting in the space $L^2(\mathbb{R}_+, \mathfrak{H})$. Here, α is a real parameter. We impose the condition

$$\int_{\mathbb{R}_+} \|V(x)\|^2\, dx < \infty. \tag{2}$$

The domain of H consists of $W_0^2(\mathbb{R}_+, \mathfrak{H})$-functions. This class of functions can be viewed as the countably infinite orthogonal sum of Sobolev spaces $W_0^2(\mathbb{R}_+)$. Besides having square integrable generalized derivatives of the second order, the $W_0^2(\mathbb{R}_+)$-functions vanish at $x = 0$.

Definition. We say that an essential support of the absolutely continuous spectrum of the operator H contains $[0, \infty)$ if the spectral projection $E_\alpha(\Omega)$ of H corresponding to any Borel set $\Omega \subset [0, \infty)$ is different from zero, $E_\alpha(\Omega) \neq 0$, as soon as the Lebesgue measure of Ω is positive.

Operators with square integrable potentials were studied by P. Deift and R. Killip [1] in the case where $\mathfrak{H} = \mathbb{C}$. The main result of [1] states that the absolutely continuous spectrum of the operator $-d^2/dx^2 + V$ covers the positive half-line $[0, \infty)$ if $V \in L^2(\mathbb{R}_+)$.

We consider the case where the space $\mathfrak{H}$ is infinite dimensional and give a different proof of the following theorem by S. Denisov [4].

Theorem 1.1. *Let V satisfy condition* (2). *Then an essential support of the absolutely continuous spectrum of the operator* (1) *contains* $[0, \infty)$ *for almost every* $\alpha \in \mathbb{R}$.

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Besides the article [4], one can also find a close discussion of similar operator families in the papers [7] and [8]. In all mentioned publications, the properties of the absolutely continuous spectrum are established for almost every value of the real parameter α. However, if $\|V(x)\| \le C(1+|x|)^{-2/3-\delta}$ with $\delta > 0$, then the absolutely continuous spectrum fills the positive half-line $\mathbb{R}_+$ for all α (see [5]). Instead of using hyperbolic pencils considered in [4], we obtain Theorem 1.1 by an application of Lemma 2.1.

§2. Auxiliary lemma

Notation. Throughout the text, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of a complex number z. For a selfadjoint operator $B = B^*$ and a vector g of a Hilbert space, the expression $((B-k-i0)^{-1}g,g)$ is always understood as the limit

$$\Big((B-k-i0)^{-1}g,g\Big) = \lim_{\varepsilon\to 0}\Big((B-k-i\varepsilon)^{-1}g,g\Big), \qquad \varepsilon>0, \quad k\in\mathbb{R}.$$

The following simple statement plays a very important role in our proof.

Lemma 2.1. *Let B be a selfadjoint operator in a separable Hilbert space $\mathfrak{H}$ and let $g\in\mathfrak{H}$. Then the function*

$$\eta(k) := \operatorname{Im}\Big((B-k-i0)^{-1}g,g\Big) \ge 0$$

is integrable over $\mathbb{R}$. Moreover,

$$\int_{-\infty}^{\infty}\eta(k)\,dk \le \pi\|g\|^2$$

and

$$\int_{-\infty}^{\infty}\frac{\eta(k)}{k^2+1}\,dk \le \pi\|(B^2+I)^{-1/2}g\|^2.$$

Proof. Let $E_B(\,\cdot\,)$ be the spectral measure of the operator B. Then

$$\Big((B-z)^{-1}g,g\Big) = \int_{\mathbb{R}}(t-z)^{-1}\,d\big(E_B(-\infty,t)g,g\big), \quad z\in\mathbb{C}\setminus\mathbb{R}.$$

Therefore, according to the Stieltjes–Perron inversion formula,

$$\pi^{-1}\eta(k) = \frac{d}{dk}\big(E_B(-\infty,k)g,g\big), \quad \text{for almost every } k\in\mathbb{R}.$$

Consequently, for any nonnegative measurable function f on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(k)\eta(k)\,dk \le \pi\int_{\mathbb{R}} f(k)\,d\big(E_B(-\infty,k)g,g\big) = \pi\big(f(B)g,g\big).$$ □

§3. Entropy

Let μ be a nonegative finite Borel measure on the real line $\mathbb{R}$. As any other measure it is decomposed uniquely into a sum of three terms

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc},$$

where the first term is pure point, the second term is absolutely continuous, and the last term is a continuous but singular measure on $\mathbb{R}$. Obviously, $\mu(-\infty,\lambda)$ is a monotone function of λ, therefore, it is differentiable almost everywhere. In particular, the limit

$$\mu'(\lambda) = \lim_{\epsilon\to 0}\frac{\mu(\lambda-\epsilon,\lambda+\epsilon)}{2\epsilon}$$

exists for almost every $\lambda\in\mathbb{R}$. It is also clear that

$$\mu_{ac}(\Omega) = \int_{\Omega}\mu'(\lambda)\,d\lambda, \quad \Omega\subset\mathbb{R},$$

which means that $\mu' = \mu'_{ac}$.

Let $\Omega_0 = \{\lambda : \mu'(\lambda) > 0\}$. A measurable set $\Omega \subset \mathbb{R}$ is called an essential support of μ_{ac} if the Lebesgue measure of the symmetric difference

$$\Omega_0 \triangle \Omega := \Big(\Omega_0 \setminus \Omega\Big) \cup \Big(\Omega \setminus \Omega_0\Big)$$

is zero. So, an essential support of μ_{ac} coincides with the set where $\mu' > 0$ up to a set of measure zero. As we see, the study of the essential support of the a.c. part of the measure μ is reduced to the study of the set $\Omega_0 = \{\lambda : \mu'(\lambda) > 0\}$. Let Ω be a measurable set. One of the ways to show that $\mu'(\lambda) > 0$ for almost every $\lambda \in \Omega$ relies on the study of the quantity

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda.$$

Due to Jenssen's inequality, $S_\Omega < \infty$ for sets of finite Lebesgue measure $|\Omega| < \infty$. So, the entropy in this case can diverge only to the negative infinity.

But if

$$S_\Omega(\mu) > -\infty, \quad \text{while} \quad |\Omega| < \infty,$$

then

$$\mu'(\lambda) > 0 \quad \text{a.e. on } \Omega.$$

Very often one can obtain an estimate for μ' by an analytic function from below. In this case, we will use the following statement.

Proposition 3.1. *Let a function $F(\lambda) \neq 0$ be analytic in a neighborhood of an interval $[a,b] \subset \mathbb{R}$. Suppose that*

$$\mu'(\lambda) > |F(\lambda)|^2, \qquad \textit{for all } \ \lambda \in \Omega \subset [a,b]. \tag{3}$$

Then

$$S_\Omega(\mu) := \int_\Omega \log \mu'(\lambda)\, d\lambda \geq C > -\infty,$$

where the constant $C = C([a,b], F)$ depends on the interval $[a,b]$ and the function F.

Proof. This proposition follows from the fact that the zeros of analytic functions are isolated and have finite multiplicities. □

In applications to Schrödinger operators, one often has an estimate of the form (3) for a sequence of measures μ_n that converges to μ weakly

$$\mu_n \to \mu \ \text{ weakly}.$$

In this situation, one can still derive a certain information about the limit measure μ from the information about μ_n.

Definition. Let ρ, ν be finite Borel measures on a compact Hausdorff space X. We define the entropy of ρ relative to ν by

$$S(\rho|\nu) = \begin{cases} -\infty & \text{if } \ \rho \text{ is not } \nu\text{-ac} \\ -\int_X \log(\frac{d\rho}{d\nu}) d\rho & \text{if } \ \rho \text{ is } \nu\text{-ac}. \end{cases} \tag{4}$$

Theorem 3.1 (cf. [6])**.** *The entropy $S(\rho|\nu)$ is jointly upper semi-continuous in ρ and ν with respect to the weak topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$ weakly, then*

$$S(\rho|\nu) \geq \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

Now, we use this theorem in order to prove the following statement.

Proposition 3.2. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in a neighborhood of $[a, b]$. Let μ_n be a sequence of finite nonnegative Borel measures on the real line $\mathbb{R}$ converging to μ weakly. Suppose that*

$$\mu_n'(\lambda) > |F(\lambda)|^2, \quad \textit{for all} \ \ \lambda \in \Omega_n \subset [a, b],$$

where the measurable sets Ω_n satisfy

$$\Big|[a, b] \setminus \Omega_n\Big| < b - a - \varepsilon.$$

Then $\mu'(\lambda) > 0$ on a subset of $[a, b]$ whose measure is not smaller than $b - a - \varepsilon$

Proof. We denote the characteristic function of the set Ω_n by χ_n. Since the L^2-norms of χ_n are uniformly bounded, this sequence of functions has a weakly convergent subsequence. Therefore without loss of generality, we may assume that

$$\chi_n \to \chi \ \ \text{weakly in} \ \ L^2(\mathbb{R}).$$

This, of course, implies that the corresponding measures $\chi_n d\lambda$ also converge weakly to $\chi d\lambda$. Even though $\mathbb{R}$ is not compact, we can still use Theorem 3.1 and show (see [7]) that

$$\int_{\mathbb{R}} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda)}\Big)\, \chi(\lambda)\, d\lambda \geq \limsup_{n\to\infty} \int_{\mathbb{R}} \log\Big(\frac{\mu_n'(\lambda)}{\chi_n(\lambda)}\Big)\, \chi_n(\lambda)\, d\lambda > -\infty.$$

Thus, we see that $\mu' > 0$ on the support of the function χ. However, we still need to know how big this set is. For that purpose, we first observe that

$$\int_a^b \chi(\lambda)\, d\lambda = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_n(\lambda)\, d\lambda \geq b - a - \varepsilon.$$

On the other hand, it is easy to show that $0 \leq \chi \leq 1$. Therefore, the Lebesgue measure of the support of the function χ is not smaller than $b - a - \varepsilon$. □

Since we deal with a family of operators depending on a parameter α, we also need a modification of the previous statement, suitable in the case when the measures also depend on the parameter α. Let $\mathcal{M}$ be the topological space whose elements are nonnegative Borel measures μ on $\mathbb{R}$ having the property $\mu(\mathbb{R}) = 1$. We define the topology on $\mathcal{M}$ to be the one that is induced by the weak-$*$ topology. Finally, let $\mathfrak{M}(\mathbb{R})$ be the class of continuous functions from $\mathbb{R}$ to $\mathcal{M}$. We are ready to state the following result.

Proposition 3.3. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic in a neighborhood of $[a, b]$. Let $\mu_n(\,\cdot\,, \alpha)$ be a sequence of α-dependent families of finite nonegative Borel measures on $\mathbb{R}$ converging to $\mu(\,\cdot\,, \alpha)$ weakly for every $\alpha \in \mathbb{R}$. Suppose the function $\alpha \mapsto \mu_n(\,\cdot\,, \alpha)$ belongs to $\mathfrak{M}(\mathbb{R})$ for each $n \in \mathbb{N}$. Finally, assume that the derivative of μ_n with respect to $d\lambda$ satisfies*

$$\mu_n'(\lambda, \alpha) > |F(\lambda)|^2, \quad \textit{for all} \ \ (\lambda, \alpha) \in \Omega_n \subset [a, b] \times [\alpha_1, \alpha_2],$$

where the measurable sets Ω_n obey

$$\big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega_n\big| < (b - a)(\alpha_2 - \alpha_1) - \varepsilon.$$

Then $\mu'(\lambda, \alpha) > 0$ on a subset of $[a, b] \times [\alpha_1, \alpha_2]$ whose measure is not smaller than $(b - a)(\alpha_2 - \alpha_1) - \varepsilon$.

The proof of this statement is a counterpart of the proof of the preceding proposition and it is left to the reader as an exercise. A similar statement was proved in [7].

We conclude this section with a discussion of the following simple claim.

Proposition 3.4. *Let $a < b$. Let $F(\lambda) \neq 0$ be a function analytic on a neighborhood of the interval $[a, b]$. Let $\mu(\,\cdot\,, \alpha)$ be an α-dependent family of finite nonegative measures on $\mathbb{R}$. Suppose that the derivatives of μ with respect to the Lebesgue measure $d\lambda$ satisfy the estimate*

$$\mu'(\lambda, \alpha) \geq |F(\lambda)|^2 (1 - \Psi(\lambda, \alpha)), \quad \textit{where} \quad \int_{\alpha_1}^{\alpha_2} \int_a^b |\Psi(\lambda, \alpha)| \, d\lambda \, d\alpha < \varepsilon/2.$$

Then

$$\mu'(\lambda, \alpha) \geq \frac{1}{2} |F(\lambda)|^2, \qquad \textit{for all} \quad (\lambda, \alpha) \in \Omega,$$

where the measureable set Ω obeys

$$\big| [a, b] \times [\alpha_1, \alpha_2] \setminus \Omega \big| \leq \varepsilon. \tag{5}$$

Proof. According to Chebyshev's inequality,

$$\Psi(\lambda, \alpha) \leq 1/2$$

on a set satisfying condition (5). □

§4. The case of a compactly supported V

In this section, we assume that V belongs to the class $\mathfrak{V}$ described below.

Definition. We say that a bounded measurable function V from $\mathbb{R}_+$ to the set of bounded selfadjoint operators on $\mathfrak{H}$ belongs to the class $\mathfrak{V}$ if

1) there is a bounded interval $[0, R]$ containing the support of V and such that $V(x + R/2)$ is an odd function of x:

$$V(x + R/2) = -V(-x + R/2), \quad x \in [0, R/2]; \tag{6}$$

2) the range of the operator $V(x)$ is a finite-dimensional subspace $\mathfrak{H}_0 \subset \mathfrak{H}$ which stays the same when one changes x.

Our proof of Theorem 1.1 is based on the relationship between the derivative of the spectral measure and the so called scattering amplitude. Both objects should be introduced properly. While the spectral measure can be defined for any selfadjoint operator, the scattering coefficient will be introduced only for a Schrödinger operator. Let f be a square integrable function from $\mathbb{R}_+$ to $\mathfrak{H}$. It is very well known that the quadratic form of the resolvent of H can be written as a Cauchy integral

$$\big((H - z)^{-1} f, f\big) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t - z}, \quad \operatorname{Im} z \neq 0.$$

The measure μ in this representation is called the spectral measure of H corresponding to the element f.

Let us introduce the scattering amplitude. Since the support of the potential V is compact, there exists an R such that $V(x) = 0$ for $x > R$. Take any bounded compactly supported function f that also vanishes for $x > R$. Then

$$\begin{aligned} &\big[(H - z)^{-1} f\big](x) = e^{ik|x|} A_f(k), \\ &\text{for} \quad x > R, \quad k^2 = z, \quad \operatorname{Im} k \geq 0, \; A_f(k) \in \mathfrak{H}. \end{aligned} \tag{7}$$

Clearly, the relation

$$\mu'(\lambda) = \pi^{-1} \lim_{z \to \lambda + i0} \operatorname{Im} \big((H - z)^{-1} f, f\big) = \pi^{-1} \lim_{z \to \lambda + i0} \operatorname{Im} z \|(H - z)^{-1} f\|^2 \tag{8}$$

implies the formula

$$\pi \mu'(\lambda) = \sqrt{\lambda} \|A_f(k)\|^2, \quad k^2 = \lambda > 0. \tag{9}$$

To prove (9), define χ_X to be the characteristic function of a set $X \subset \mathbb{R}_+$. Since the limit

$$\lim_{z\to\lambda+i0} \|\chi_{[0,b]}(H-z)^{-1}f\|^2$$

(along the vertical directions) exists and is finite for each $b > 0$, we infer from (8) that

$$\mu'(\lambda) = \pi^{-1} \lim_{z\to\lambda+i0} \mathrm{Im}\ z\|\chi_{[R,\infty)}(H-z)^{-1}f\|^2.$$

Now (9) follows by (7), because

$$\|\chi_{[R,\infty)}(H-z)^{-1}f\|^2 = \frac{e^{-2\,\mathrm{Im}\ kR}}{2\,\mathrm{Im}\ k}\|A_f(k)\|^2, \qquad \text{for } \mathrm{Im}\ k > 0.$$

The remaining arguments in this paper will be devoted to a lower estimate of $\|A_f(k)\|$.

For our purposes, it is sufficient to assume that f is the product of the characteristic function of the unit interval $[0, 1]$ times a unit vector $\tau \in \mathfrak{H}$. Traditionally, H is viewed as an operator obtained by a perturbation of

$$H_0 = -\frac{d^2}{dx^2}.$$

In its turn, $(H - z)^{-1}$ can be viewed as an operator obtained by a perturbation of $(H_0 - z)^{-1}$. The theory of such perturbations is often based on the second resolvent identity

$$(H-z)^{-1} = (H_0-z)^{-1} - (H-z)^{-1}\,\alpha V(H_0-z)^{-1}, \tag{10}$$

which turns out to be useful for our arguments. As a consequence of (10), we obtain

$$A_f(k) = F_0(k)\tau - A_g(k), \quad z = k^2 + i0,\ k > 0, \tag{11}$$

where $g(x) = \alpha V(H_0 - z)^{-1}f$ and the number $F_0(k) \in \mathbb{C}$ is defined by

$$(H_0-z)^{-1}f = e^{ik|x|}F_0(k)\tau, \quad \text{for } x > 1. \tag{12}$$

We will shortly show that, without loss of generality, we may assume that $V(x)\tau = 0$ inside the unit interval $[0, 1]$. In this case,

$$g = F_0(k)h_k, \quad \text{where} \quad h_k(x) = \alpha e^{ik|x|}V\tau. \tag{13}$$

According to (11),

$$2\|A_f(k)\|^2 \ge |F_0(k)|^2 - 2\|A_g(k)\|^2,$$

which can be written in the form

$$2\pi\mu'(\lambda) \ge |F_0(k)|^2\Big(\sqrt{\lambda} - 2\ \mathrm{Im}\ \Big((H-z)^{-1}h_k,\, h_k\Big)\Big), \quad z = \lambda + i0, \tag{14}$$

due to (9) and (13). Therefore, in order to establish the presence of the absolutely continuous spectrum, we need to show that the quantity

$$\mathrm{Im}\ \big((H-z)^{-1}h_k,\, h_k\big)$$

is small.

Let us define η setting

$$\alpha^2k^{-2}\eta(k,\alpha) := \frac{1}{k}\,\mathrm{Im}\Big((H-z)^{-1}h_k,\, h_k\Big) \ge 0, \quad z = k^2 + i0.$$

Obviously, η is positive for all real $k \ne 0$, because we agreed that $z = k^2 \pm i0$ if $\pm k > 0$. This is very convenient. Since $\eta \ge 0$, we can conclude that η is small on a rather large set if the integral of this function is small. That is why we will estimate

$$J(V) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta(k,\alpha)}{(\alpha^2+k^2)}\,\frac{|k|\,dk\,d\alpha}{(k^2+1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta(k,tk)}{(k^2+1)(t^2+1)}\,dk\,dt. \tag{15}$$

We will employ a couple of tricks, one of which is related to the involvment of an additional parameter ε. Instead of dealing with the operator H, we will deal with $H+\varepsilon I$ where $\varepsilon>0$ is small. We will first obtain an integral estimate for the quantity

$$\eta_\varepsilon(k,\alpha)=\frac{k}{\alpha^2}\operatorname{Im}\Big((H+\varepsilon-z)^{-1}h_k,\,h_k\Big),\quad z=k^2+i0.$$

Then, since

$$\eta(k,\alpha)=\lim_{\varepsilon\to0}\eta_\varepsilon(k,\alpha)\qquad\text{a.e. on } \mathbb{R}\times\mathbb{R},$$

we conclude by Fatou's lemma that

$$J(V)\le\liminf_{\varepsilon\to0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,\alpha)}{(\alpha^2+k^2)}\frac{|k|\,dk\,d\alpha}{(k^2+1)}.$$

The second trick is to set $\alpha=kt$ and represent η_ε in the form

$$\eta_\varepsilon(k,kt)=\operatorname{Im}\Big((B+1/k-i0)^{-1}H_\varepsilon^{-1/2}v,\,H_\varepsilon^{-1/2}v\Big)\tag{16}$$

where $v=V\tau$, $H_\varepsilon=-d^2/dx^2+\varepsilon I$, and B is the bounded selfadjoint operator defined by

$$B=H_\varepsilon^{-1/2}\Big(-2i\frac{d}{dx}+tV\Big)H_\varepsilon^{-1/2}.$$

This operator is bounded, as $H_\varepsilon^{-1/2}$ is a continuous map from $L^2(\mathbb{R}_+,\mathfrak{H})$ to $W_0^1(\mathbb{R}_+,\mathfrak{H})$, while the middle factor $(-2i\frac{d}{dx}+tV)$ is a continuous map from $W_0^1(\mathbb{R}_+,\mathfrak{H})$ to $L^2(\mathbb{R}_+,\mathfrak{H})$. Since the quadratic form of the operator B is real, this operator is symmetric, and hence it is selfadjoint.

In order to justify (16) at least formally, one has to introduce the operator U of multiplication by the function $\exp(ikx)$. Using this notation, we can represent η_ε in the following form

$$\eta_\varepsilon(k,tk)=k\operatorname{Im}\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,\,v\Big),\quad z=k^2+i0.$$

Since we deal with unitary equivalence of operators, we can employ the formula

$$\Big(U^{-1}(H+\varepsilon-z)^{-1}Uv,v\Big)=\Big((U^{-1}HU+\varepsilon-z)^{-1}v,v\Big),\quad z=k^2+i0.$$

On the other hand, since H is a differential operator and U is an operator of multiplication, the commutator $[H,U]:=HU-UH$ can easily be found:

$$\Big[H,U\Big]=kU\Big(-2i\frac{d}{dx}+k\Big)\Big|_{D(H)}\quad\text{on}\ \ D(H).\tag{17}$$

Using the formula $U^{-1}HU=H+U^{-1}[H,U]$, we infer from (17) that

$$U^{-1}HU+\varepsilon-z=H_\varepsilon+k\Big(-2i\frac{d}{dx}+tV\Big)=H_\varepsilon^{1/2}(I+kB)H_\varepsilon^{1/2}.$$

If $\widetilde{k}$ belongs to the upper half-plane, then so does $-1/\widetilde{k}$. Consequently,

$$k\big(U^{-1}(H+\varepsilon-z)^{-1}Uv,v\big)=\big(H_\varepsilon^{-1/2}(B+1/k-i0)^{-1}H_\varepsilon^{-1/2}v,v\big),\quad z=k^2+i0.\tag{18}$$

In fact, (18) holds true for $\operatorname{Im}k>0$ when U is not a unitary operator, but we only need it for $k\in\mathbb{R}$.

Since B is a selfadjoint operator, $\pi^{-1}\eta_\varepsilon(k,kt)$ coincides with the derivative of the spectral measure of the operator B corresponding to the element $H_\varepsilon^{-1/2}v$. According to Lemma 2.1, the last observation implies that

$$\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,kt)}{(1+k^2)}\,dk\le\pi\Big((B^2+I)^{-1}H_\varepsilon^{-1/2}v,H_\varepsilon^{-1/2}v\Big),$$

which leads to

$$\int_{-\infty}^{\infty} \frac{\eta_\varepsilon(k,kt)}{(1+k^2)}\,dk \le \pi\Big(B^{-1}H_\varepsilon^{-1/2}v, B^{-1}H_\varepsilon^{-1/2}v\Big) = \pi\|B^{-1}H_\varepsilon^{-1/2}v\|^2, \tag{19}$$

provided B is invertible. Our further arguments will be related to the estimate of the quantity on the right-hand side of (19). We will show that B has an unbounded inverse with the property

$$\begin{gathered}\lim_{\varepsilon\to 0}\|B^{-1}H_\varepsilon^{-1/2}v\|^2 \le \int_{\mathbb{R}_+}\|V(x)\|^2\,dx,\\ D(B^{-1}) = \mathrm{Ran}\,(B) \subset W_0^1(\mathbb{R}_+,\mathfrak{H}).\end{gathered} \tag{20}$$

Our proof of (20) is based on the representation

$$B^{-1}H_\varepsilon^{-1/2}v = H_\varepsilon^{1/2}T^{-1}v, \tag{21}$$

where $T\subset T^*$ is the first order differential (symmetric) operator defined by

$$T = -2i\frac{d}{dx} + tV, \quad D(T) = D(H_\varepsilon^{1/2}) = W_0^1(\mathbb{R}_+,\mathfrak{H}).$$

As we will see, $H_\varepsilon^{-1/2}$ is a one-to-one mapping of $D(T^{-1})$ onto $D(B^{-1})$, and (21) is true for all $v\in D(T^{-1})$. To establish (21), observe that the formula $B = H_\varepsilon^{-1/2}TH_\varepsilon^{-1/2}$ leads to the relations $\mathrm{Ran}(B)\subset D(H_\varepsilon^{1/2})$ and $H_\varepsilon^{1/2}B = TH_\varepsilon^{-1/2}$. The latter of the two relations clearly implies (21) provided T is invertible and $v\in D(T^{-1})$.

Indeed, any $v\in D(T^{-1})$ can be written in the form $v = Tw$ where $w\in D(T) = D(H_\varepsilon^{1/2})$. Consequently, there is a unique vector $u\in\mathfrak{H}$ for which $v = TH_\varepsilon^{-1/2}u = H_\varepsilon^{1/2}B\,u$. Therefore $H_\varepsilon^{-1/2}v\in\mathrm{Ran}\,(B)$ and both sides of (21) coincide with the vector u.

Obviously, B is invertible if T is invertible. On the other hand, one can establish the invertibility of T by deriving an explicit formula for T^{-1} (which is also an unbounded operator). For that purpose we define U_0 to be the unitary operator of multiplication by the solution of the differential equation

$$\frac{d}{dx}U_0(x) = \frac{it}{2}U_0(x)V(x), \quad U_0(0) = I.$$

The object on the right-hand side is the composition of two operators in $\mathfrak{H}$. The solution of this differential equation exists on all of $\mathbb{R}_+$ because the equation is linear and $V\in\mathfrak{V}$. Now we see that

$$T = -2iU_0^{-1}\Big[\frac{d}{dx}\Big]U_0, \quad \text{and} \quad T^{-1} = \frac{i}{2}U_0^{-1}\Big[\frac{d}{dx}\Big]^{-1}U_0.$$

Since $[\frac{d}{dx}]^{-1}$ is precisely the simple integration with respect to x and

$$\frac{d}{dx}U_0\tau = \frac{i}{2}tU_0V\tau$$

we obtain

$$\begin{aligned}\big[T^{-1}v\big](x) &= \frac{i}{2}U_0^{-1}(x)\int_0^x U_0(y)V(y)\tau\,dy\\ &= \frac{1}{t}U_0^{-1}(x)(U_0(x)-I)\tau = \frac{1}{t}(I-U_0^{-1}(x))\tau.\end{aligned} \tag{22}$$

Note that due to condition (6), the function $T^{-1}v$ is compactly supported, which leaves no doubt about the relation $v\in D(T^{-1})$. Combining (21) with (22) and using the fact that $\|H_\varepsilon^{1/2}u\|^2 = \|\frac{d}{dx}u\|^2 + \varepsilon\|u\|^2$ for all $u\in D(H_\varepsilon^{1/2})$, we conclude that

$$\lim_{\varepsilon\to 0}\|B^{-1}H_\varepsilon^{-1/2}v\|^2 = \lim_{\varepsilon\to 0}\|H_\varepsilon^{1/2}T^{-1}v\|^2 = \int_{\mathbb{R}_+}\|V(x)U_0^{-1}(x)\tau\|^2\,dx. \tag{23}$$

Thus, (20) is established. Relations (19), (20) lead to the inequality

$$J(V) \leq \pi^2 \int_{\mathbb{R}_+} \|V(x)\|^2\,dx,$$

where the quantity $J(V)$ is from (15). However, we can say more.

Lemma 4.1. *Let $T > 0$. Let V be a potential of the class $\mathfrak{V}$ such that*

$$V(x)\tau = 0, \quad \textit{for all} \quad x < T. \tag{24}$$

Then

$$J(V) \leq \pi^2 \int_T^\infty \|V(x)\|^2\,dx. \tag{25}$$

Proof. If (24) holds, then $U_0(x)\tau = \tau$ for all $x < T$. Therefore, the right-hand side of (23) can be estimated as follows

$$\int_{\mathbb{R}_+} \|V(x)U_0^{-1}(x)\tau\|^2\,dx \leq \int_T^\infty \|V(x)\|^2\,dx. \qquad \square$$

§5. Approximations of potentials and spectral measures

Proposition 5.1. *Let $T > 0$. Let $\widetilde{V}$ be the potential*

$$\widetilde{V}(x) = V(x) - \langle\,\cdot\,,\tau\rangle V(x)\tau - \langle\,\cdot\,,V(x)\tau\rangle\tau + \langle V(x)\tau,\tau\rangle\langle\,\cdot\,,\tau\rangle\tau, \qquad \textit{for all} \quad x < T, \tag{26}$$

and let

$$\widetilde{V}(x) = V(x), \quad \textit{for all} \quad x > T. \tag{27}$$

Then

$$\Bigl(H - z\Bigr)^{-1} - \Bigl(-\frac{d^2}{dx^2} + \alpha\widetilde{V} - z\Bigr)^{-1} \in \mathfrak{S}_1 \tag{28}$$

is a trace class operator for any z with $\operatorname{Im} z > 0$.

Proof. Using Hilbert's identity, we obtain

$$\Bigl(H - z\Bigr)^{-1} - \Bigl(-\frac{d^2}{dx^2} + \alpha\widetilde{V} - z\Bigr)^{-1} = \alpha\Bigl(H - z\Bigr)^{-1}(\widetilde{V} - V)\Bigl(-\frac{d^2}{dx^2} + \alpha\widetilde{V} - z\Bigr)^{-1}.$$

Consequently, it is sufficient to prove that

$$\Gamma := \Bigl(-\frac{d^2}{dx^2} - z\Bigr)^{-1}(\widetilde{V} - V)\Bigl(-\frac{d^2}{dx^2} - z\Bigr)^{-1} \in \mathfrak{S}_1.$$

Observe now that $\widetilde{V}(x) - V(x)$ is a finite rank operator of the form

$$\widetilde{V}(x) - V(x) = w_1(x)\langle\,\cdot\,,e_1(x)\rangle e_1(x) + w_2(x)\langle\,\cdot\,,e_2(x)\rangle e_2(x),,$$

where the $w_j \in L^1(\mathbb{R}_+)$ are real valued compactly supported functions and the $e_j(x)$ are unit vectors in $\mathfrak{H}$. Since $(-\frac{d^2}{dx^2} - z)^{-1}$ is an integral operator whose integral kernel $r(x,y)$ satisfies

$$\sup_x \int_0^\infty |r(x,y)|^2\,dy + \sup_y \int_0^\infty |r(x,y)|^2\,dx < \infty,$$

the operators $G_j(z)$ defined by

$$\bigl[G_j(z)u\bigr](x) = \int_0^\infty |w_j(x)|^{1/2}\langle r(x,y)u(y), e_j(x)\rangle e_j\,dy$$

are Hilbert–Schmidt operators. It remains to note that

$$\Gamma = G_1^*(\bar z)\Omega_1 G_1(z) + G_2^*(\bar z)\Omega_2 G_2(z)$$

where the Ω_j are bounded. $\square$

According to Birman's theorem (see [2, 3]), we can now state the following result.

Proposition 5.2. *Let $\widetilde{V}$ be defined as in (26). Then the absolutely continuous parts of the operators H and $-\frac{d^2}{dx^2}+\alpha\widetilde{V}$ are unitarily equivalent.*

Let $\delta>0$. The proposition allows one to assume that there is a $T>0$ having the following properties:

1) $V(x)\tau=0$ for all $x<T$.

2) the value of the integral $\int_T^\infty \|V(x)\|^2\,dx$ is smaller than δ;

If that is not true, we replace V by $\widetilde{V}$ defined by (26) for a sufficiently large $T>0$.

Now we use inequality (14) and employ Proposition 3.4 with

$$F(\lambda)=(2\pi)^{-1/2}F_0(\sqrt{\lambda})\lambda^{1/4} \quad \text{and} \quad \Psi(\lambda)=\frac{2\,\mathrm{Im}((H-z)^{-1}h_k,h_k)}{\sqrt{\lambda}}.$$

According to Lemma 4.1, we obtain the following result.

Theorem 5.1. *Let $0<a<b<\infty$, let $0<\alpha_1<\alpha_2<\infty$, and let $T>1$. For any $\varepsilon>0$ there is a number $\delta>0$ such that for any potential V in the class $\mathfrak{V}$ having the properties*

$$1)\quad V(x)\tau=0 \quad \textit{for all} \quad x<T, \quad \textit{and} \quad 2)\quad \int_T^\infty \|V(x)\|^2\,dx<\delta,$$

the derivative $\mu'(\lambda)=\mu'(\lambda,\alpha)$ of the spectral measure satisfies the inequality

$$\mu'(\lambda,\alpha)\geq(4\pi)^{-1}|F_0(\sqrt{\lambda})|^2\lambda^{1/2}, \quad \textit{for all} \quad (\lambda,\alpha)\in\Omega,$$

where the measurable set Ω obeys

$$\big|[a,b]\times[\alpha_1,\alpha_2]\setminus\Omega\big|\leq\varepsilon.$$

The proof of the next statement is left to the reader as an exercise. A function V from $\mathbb{R}_+$ to the class of bounded operators on $\mathfrak{H}$ is said to be measurable provided the function $x\mapsto\langle V(x)h,h\rangle$ is measurable for each $h\in\mathfrak{H}$.

Proposition 5.3. *Let V be a measurable operator-valued function obeying*

$$\int_{\mathbb{R}_+}\|V(x)\|^2\,dx<\infty.$$

Assume that

$$V(x)\tau=0, \qquad \textit{for all} \quad x<T, \tag{29}$$

where $T>0$ is a fixed number. Then there is a sequence of compactly supported operator-valued functions $V_n\in\mathfrak{V}$ having the following three properties:

1)

$$V_n(x)\tau=0, \quad \textit{for all} \quad x<T,$$

2)

$$\int_T^\infty\|V_n(x)\|^2\,dx\leq 2\int_T^\infty\|V(x)\|^2\,dx, \tag{30}$$

and

3)

$$\int_0^K\|\big(V_n(x)-V(x)\big)u(x)\|^2\,dx\to 0, \quad \textit{as} \quad n\to\infty,$$
$$\textit{for any } u\in L^\infty(\mathbb{R}_+,\mathfrak{H}) \quad \textit{and any} \quad K>0.$$

Another statement, which we are going to use, deals with the spectral measures of operators whose potentials V_n approximate the function V.

Proposition 5.4. *Let $V \in L^2(\mathbb{R}_+, \mathfrak{H})$ and $V_n \in L^2(\mathbb{R}_+, \mathfrak{H})$ obey (30) for some $T > 0$. Let μ_n and μ be the spectral measures of the operators H_n and H with potentials αV_n and αV, correspondingly. Assume that*

$$\int_0^K \|\big(V_n(x) - V(x)\big)u(x)\|^2\,dx \to 0, \quad \text{as} \quad n \to \infty,$$

$$\text{for any } u \in L^\infty(\mathbb{R}_+, \mathfrak{H}) \quad \text{and any } K > 0.$$

Then

$$\mu_n \to \mu \quad \text{weakly}, \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad \alpha \in \mathbb{R}.$$

The *proof* of this proposition is rather standard. First observe that the set of finite linear combinations of functions of the form

$$\phi_z(t) = \operatorname{Im}\big(1/(t - z)\big)$$

with $\operatorname{Im} z > 0$ is dense in the space of functions that are continuous on $\mathbb{R}$ and decay at infinity. Consequently, it suffices to show that

$$\int_{\mathbb{R}} \phi_z(t)\,d\mu_n(t) \to \int_{\mathbb{R}} \phi_z(t)\,d\mu(t) \quad \text{as} \quad n \to \infty$$

for each $z \in \mathbb{C}_+$. According to the definition of the measures μ_n and μ, this is the same as showing that

$$\operatorname{Im}\big((H_n - z)^{-1}f, f\big) \to \operatorname{Im}\big((H - z)^{-1}f, f\big), \quad \text{as} \quad n \to \infty.$$

The last property follows from the identity

$$\big((H_n - z)^{-1}f, f\big) - \big((H - z)^{-1}f, f\big) = \big((H_n - z)^{-1}(V - V_n)(H - z)^{-1}f, f\big),$$

because the condition $(H - z)^{-1}f \in W_0^1(\mathbb{R}_+, \mathfrak{H})$ implies that

$$\|(V - V_n)(H - z)^{-1}f\| \to 0$$

as $n \to \infty$. □

According to Proposition 3.2, the assertion below follows from Theorem 5.1 combined with Propositions 5.3 and 5.4.

Theorem 5.2. *Let $0 < a < b < \infty$, let $0 < \alpha_1 < \alpha_2 < \infty$, and let $T > 1$. For any $\varepsilon > 0$ there is a number $\delta > 0$ such that for any potential $V \in L^2(\mathbb{R}_+, \mathfrak{H})$ having the properties*

$$1) \quad V(x)\tau = 0 \qquad \text{for all} \quad x < T, \qquad \text{and} \quad 2) \quad \int_T^\infty \|V(x)\|^2\,dx < \delta,$$

the derivative $\mu'(\lambda) = \mu'(\lambda, \alpha)$ of the spectral measure is positive

$$\mu'(\lambda, \alpha) > 0, \quad \text{for all} \quad (\lambda, \alpha) \in \Omega, \tag{31}$$

where the measurable set Ω obeys

$$\big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega\big| \le \varepsilon.$$

Let $E_\alpha(\,\cdot\,)$ be the operator-valued spectral measure of H. Let also

$$\Omega_\alpha = \{\lambda \in [a, b] \,:\, (\lambda, \alpha) \in \Omega\}$$

be the cross-section of Ω. One can conclude from inequality (31) that, for any measurable subset $X \subset [a, b]$, the condition $E_\alpha(X) = 0$ implies the relation

$$\big|\Omega_\alpha \cap X\big| = 0.$$

Using the unitary equivalence claimed by Proposition 5.2, we obtain the following statement.

Theorem 5.3. *Let $0 < a < b < \infty$, let $0 < \alpha_1 < \alpha_2 < \infty$. Assume that $V \in L^2(\mathbb{R}_+, \mathfrak{H})$. Then for any $\varepsilon > 0$, there is a measurable set $\Omega(\varepsilon) \subset [a, b] \times [\alpha_1, \alpha_2]$ obeying*

$$\big|[a, b] \times [\alpha_1, \alpha_2] \setminus \Omega(\varepsilon)\big| \le \varepsilon$$

such that, for any Borel set $X \subset [a, b]$ and the cross-section $\Omega_\alpha(\varepsilon)$ defined by

$$\Omega_\alpha(\varepsilon) = \{\lambda \in [a, b] \,:\, (\lambda, \alpha) \in \Omega(\varepsilon)\},$$

the condition $E_\alpha(X) = 0$ implies the relation

$$\big|\Omega_\alpha(\varepsilon) \cap X\big| = 0.$$

Take now a monotonically decreasing sequence ε_n converging to 0, as $n \to \infty$, and set

$$\widetilde{\Omega} = \bigcup_{n=1}^{\infty} \Omega(\varepsilon_n).$$

Obviously, $\widetilde{\Omega}$ is a subset of full measure in $[a, b] \times [\alpha_1, \alpha_2]$. Consequently,

$$\widetilde{\Omega}_\alpha = \{\lambda \in [a, b] \,:\, (\lambda, \alpha) \in \widetilde{\Omega}\}$$

is a subset of full measure in $[a, b]$ for almost every $\alpha \in [\alpha_1, \alpha_2]$.

Take now an arbitrary Borel subset $X \subset [a, b]$. If $|X \cap \widetilde{\Omega}_\alpha| > 0$, then there is an integer n for which

$$\big|\Omega_\alpha(\varepsilon_n) \cap X\big| > 0.$$

This condition implies that $E_\alpha(X) \ne 0$. Thus, the essential support of the absolutely continuous spectrum of H contains the interval $[a, b]$ for all α such that

$$|\widetilde{\Omega}_\alpha| = b - a. \tag{32}$$

It remains to note that (32) holds for almost every $\alpha \in [\alpha_1, \alpha_2]$.

This completes the proof of Theorem 1.1. □

References

[1] P. Deift and R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials*, Comm. Math. Phys. **203** (1999), no. 2, 341–347. MR 1697600

[2] M. Sh. Birman, *Conditions for existence of wave operators*, Dokl. Akad. Nauk SSSR **143** (1962), no. 3, 506–509; English transl., Soviet Math. Dokl. **3** (1962), 408–411. MR 0149333

[3] ———, *Existence conditions for wave operators*, Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), no. 4, 883–906; English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 54, Amer. Math. Soc., Providence, RI, 1966, pp. 91–117. MR 0161150

[4] S. Denisov, *Schrödinger operators and associated hyperbolic pencils*, J. Funct. Anal. **254** (2008), no. 8, 2186–2226. MR 2402106

[5] ———, *Spatial asymptotics of Green's function and applications*, J. Anal. Math. **148** (2022), no. 2, 501–522. MR 4525761

[6] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. (2) **158** (2003), no. 1, 253–321. MR 1999923

[7] O. Safronov, *Absolutely continuous spectrum of a one-parameter family of Schrödinger operators*, Algebra i Analiz **24** (2012), no. 6, 178-195; English transl., St. Petersburg Math. J. **24** (2013), no. 6, 977-989. MR 3097557

[8] ———, *Absolutely continuous spectrum of a typical Schrödinger operator with a slowly decaying potential*, Proc. Amer. Math. Soc. **142** (2014), no. 2, 639–649. MR 3134004

[9] B. Vainberg, *On the analytical properties of the resolvent for a certain class of operator-pencils*, Sb. Math. **6** (1968), no. 2, 259–296; English transl., Math. USSR-Sb. **6** (1968), no. 2, 241–273. MR 0240447

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