LIEB–THIRRING INEQUALITIES ON MANIFOLDS WITH CONSTANT NEGATIVE CURVATURE

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Abstract. In this short note we prove Lieb–Thirring inequalities on manifolds with negative constant curvature. The discrete spectrum appears below the continuous spectrum $[(d-1)^2/4, \infty)$, where d is the dimension of the hyperbolic space. As an application we obtain a Pólya type inequality with not a sharp constant. An example of a 2D domain is given for which numerical calculations suggest that the Pólya inequality holds for it.

1. INTRODUCTION

Lieb–Thirring inequalities have important applications in mathematical physics, analysis, dynamical systems and attractors, to mention a few. A current state of the art of many aspects of the theory is presented in [11]. We mention here the celebrated paper by Lieb and Thirring [23], where such inequalities were studied for the questions of stability of matter.

In certain applications Lieb–Thirring inequalities are considered on a manifold. For example, on torus or sphere one has to impose the zero mean orthogonality condition, since the Laplacian has a simple zero eigenvalue corresponding to a constant function, see [12], [15], [29], [14], [13]. Such inequalities are useful in the study of the dimension of attractors in theory of Navier-Stokes equation.

In this work we prove Lieb–Thirring inequalities on manifolds with negative constant curvature. Let $\mathbb{H}^d$, $d \geq 2$, be the open upper half-space

$$\mathbb{H}^d = \{(x, y) : \, x \in \mathbb{R}^{d-1}, \, y > 0\}$$

with the Poincare metric $ds^2 = y^{-2}(dx^2 + dy^2)$. We consider the self-adjoint Laplace–Beltrami operator in $L^2\left(\mathbb{H}^d, \frac{dx\, dy}{y^d}\right)$

$$-\Delta_h = -y^d \frac{\partial}{\partial y}\, y^{2-d}\, \frac{\partial}{\partial y} - y^2 \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2}, \tag{1.1}$$

defined by the lower semi-bounded quadratic form.

The spectrum of the standard Laplacian

$$-\Delta = -\sum_{n=1}^{d} \frac{\partial^2}{\partial x_n^2}$$

acting in $L^2(\mathbb{R}^d, dx)$ is absolutely continuous and covers the whole half-line $[0, \infty)$. By contrast, the spectrum of the Laplace operator (1.1) is absolutely continuous and covers the interval $[(d-1)^2/4, \infty)$, see, for example, [24], [25], [18].

Denote by $\mathrm{L}^{\mathrm{cl}}_{\gamma,d}$ the value

$$\mathrm{L}^{\mathrm{cl}}_{\gamma,d} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|^2)^\gamma_+ \, d\xi = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma + d/2 + 1)}.$$

Let $V = V(x) \ge 0$ and let $V \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then the Schrödinger operator $-\Delta - V$ is well defined as a lower semi-bounded self-adjoint operator on $L^2(\mathbb{R}^d)$ and the classical Lieb–Thirring inequality states that

$$\sum \nu_k^\gamma = \mathrm{Tr}\, (-\Delta - V)^\gamma_- \le \mathrm{L}_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma+d/2} \, dx,$$

where $-\nu_k \le 0$ are the negative eigenvalues of the Schrödinder operator $-\Delta - V$.

In [23] E.H. Lieb and W. Thirring proved that it holds for finite explicitly given constants $\mathrm{L}_{\gamma,d}$ as long as $\gamma > \max(0, 1 - d/2)$. In the case when $d \ge 3$, $\gamma = 0$ this bound is known as the Cwikel–Lieb–Rozenblum (CLR) inequality, see [4, 22, 27]. The second critical case $d = 1, \gamma = 1/2$ was settled by Weidl in [30]. The inequality is known to fail for $d = 2$, and $\gamma = 0$. Sharp constants are known for $\gamma \ge 3/2$ and $\gamma = 1/2$, $d = 1$:

$$\mathrm{L}_{\gamma,d} = \mathrm{L}^{\mathrm{cl}}_{\gamma,d},\ \gamma \ge 3/2,\ d \ge 1 \qquad \text{and} \qquad \mathrm{L}_{1/2,1} = 2\mathrm{L}^{\mathrm{cl}}_{1/2,1}, \tag{1.2}$$

see [17] and [8, 9], respectively. In the remaining cases one has

$$\mathrm{L}_{\gamma,d} \le \begin{cases} 2\mathrm{L}^{\mathrm{cl}}_{\gamma,d}, & 1/2 \le \gamma,\ d = 1 \\ R_{1,1}\mathrm{L}^{\mathrm{cl}}_{\gamma,d}, & 1 \le \gamma < 3/2, \\ 2R_{1,1}\mathrm{L}^{\mathrm{cl}}_{\gamma,d}, & 1/2 \le \gamma < 1,\ d \ge 2. \end{cases} \tag{1.3}$$

The value $R_{1,1}$ here, $R_{1,1} \le 1.456\ldots$, was obtained in the recent paper [10] on the new bounds for the constants in the Lieb–Thirring inequality for the one-dimensional Schrödinger operator with an operator-valued potential.

For a comprehensive treatment and references of the subject, see [11].

In this paper we study the spectrum of the Schrödinger operator

$$-\Delta_h - V. \tag{1.4}$$

Here $V \in L^{\gamma+d/2}\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$, which makes it possible to define this operator as a lower semi-bounded self-adjoint operator in $L^2\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$. We obtain Lieb–Thirring inequalities for the discrete spectrum below $(d-1)^2/4$.

It is convenient to denote the eigenvalues $\{\lambda_k\}$ of the operator (1.4) in terms of the negative values $\{-\mu_k\}$, where

$$\lambda_k = \frac{(d-1)^2}{4} - \mu_k. \tag{1.5}$$

To the best of our knowledge the Lieb–Thirring inequalities for the Schödinger operator in the hyperbolic space have never been studied before. Certain estimates for the eigenvalues of this operator in the non-Hilbert case were studied in [7]. There was a considerable interest in the literature concerning the CLR inequality in the hyperbolic metric and other non-Euclidean metrics, see [1, 19, 20] and the references therein. We also point out that the Lieb–Thirring inequalities for γ-moments with $\gamma > 0$ can in the standard way be derived from the CLR inequality. However, one has an obvious restriction $d \geq 3$ here in the first place, and, secondly, the constants obtained by this approach are much worse. For instance, even in the case of $\mathbb{R}^d$ the best known estimate for the CLR constant in $\mathbb{R}^3$ is Lieb's bound $\mathrm{L}_{0,3} \leq 6.8693 \cdot \mathrm{L}^{\mathrm{cl}}_{0,3}$ (see [22]), and this factor in the constants will propagate to higher order γ-moments with $\gamma > 0$.

The main result of the paper is the following:

Theorem 1.1. *Let $V \geq 0$ and $\gamma \geq 1/2$. Then*

$$\sum \mu_k^\gamma \leq \mathbb{L}_{\gamma,d} \int_{\mathbb{H}^d} V(x,y)^{\gamma+d/2} \, \frac{dx\,dy}{y^d}, \tag{1.6}$$

where the constant $\mathbb{L}_{\gamma,d}$ satisfies

$$\mathbb{L}_{\gamma,d} \leq \begin{cases} \mathrm{L}^{\mathrm{cl}}_{\gamma,d}, & 3/2 \leq \gamma, \\ R_{1,1}\mathrm{L}^{\mathrm{cl}}_{\gamma,d}, & 1 \leq \gamma < 3/2, \\ 2R_{1,1}\mathrm{L}^{\mathrm{cl}}_{\gamma,d}, & 1/2 \leq \gamma < 1. \end{cases} \tag{1.7}$$

Remark 1.1. *Concerning the constant $\mathbb{L}_{\gamma,d}$, this theorem essentially says that*

$$\mathbb{L}_{\gamma,d} \leq \mathrm{L}_{\gamma,d}$$

if we interpret $\mathrm{L}_{\gamma,d}$ as the best known to date constant in the classical Lieb–Thirring inequality.

Remark 1.2. *We do not claim that our estimate of the constant $\mathbb{L}_{\gamma,d} \leq \mathrm{L}^{\mathrm{cl}}_{\gamma,d}$ for $\gamma \geq 3/2$ in* (1.7) *is sharp. However, the following Weyl-type asymptotic formula in* [20]*:*

$$\lim_{\alpha\to\infty} \alpha^{-d/2}\mathcal{N}\big((d-1)^2/4, -\Delta_h - \alpha V\big) = \mathrm{L}^{\mathrm{cl}}_{0,d} \int_{\mathbb{H}^d} V(x,y)^{d/2} \, \frac{dx\,dy}{y^d},$$

where $d \geq 3$ and $N\big((d-1)^2/4, -\Delta_h - \alpha V\big)$ is the number of the eigenvalues below the bottom of the continuous spectrum of the operator $-\Delta_h - \alpha V$, *provides some evidence that, in fact,* $\mathbb{L}_{\gamma,d} = \mathrm{L}^{\mathrm{cl}}_{\gamma,d}$, $\gamma \geq 3/2$ *at least for* $d \geq 3$. *We shall not go into further details here.*

In the next Section 2 we present a simple proof of the fact that the continuous spectrum of the Laplacian on hyperbolic space with curvature -1 covers the semi-axis $[(d-1)^2/4, \infty)$. In Section 3 we give the proof the main Theorem 1.1 and in Section 4 we obtain the dual inequality that could be used for estimates of the dimension of attractors in theory of the Navier–Stokes equation.

In Section 5 we apply Theorem 1.1 to derive an inequality on the number of eigenvalues below $\Lambda > 0$ for Dirichlet Laplace–Beltrami operator on a domain $\Omega \subset \mathbb{H}^d$ of finite hyperbolic measure.

Assume that $\Omega \subset \overline{\Omega} \subset \mathbb{H}^d$ satisfies the inequality

$$|\Omega|_h = \int_\Omega \frac{dxdy}{y^d} < \infty.$$

We consider the Dirichlet eigenvalue problem for the Laplace–Beltrami operator $-\Delta_h$ in $L^2(\Omega, y^{-d}dxdy)$ defined via the respective quadratic form

$$-\Delta_h u = \lambda u, \qquad u\big|_{(x,y)\in\partial\Omega} = 0. \tag{1.8}$$

The spectrum of this operator is discrete and we denote by $\{\lambda_k\}$ its eigenvalues. Such eigenvalues satisfy the inequality

$$\lambda_k > \frac{(d-1)^2}{4}.$$

Similarly to (1.5) it is convenient to introduce the numbers ν_k such that

$$\lambda_k = \frac{(d-1)^2}{4} + \nu_k \tag{1.9}$$

and study the counting function $\mathcal{N}(\Lambda)$ of the spectrum

$$\mathcal{N}(\Lambda) = \#\{k : \nu_k < \Lambda\}, \qquad \Lambda > 0.$$

Theorem 1.2. *Let* $|\Omega|_h < \infty$. *Then the counting function* $\mathcal{N}(\Lambda)$ *of the eigenvalues of the spectral problem* (1.8) *satisfies the following inequality*

$$\mathcal{N}(\Lambda) \leq \left(1 + \frac{2}{d}\right)^{d/2} \left(1 + \frac{d}{2}\right) \mathbb{L}_{1,d}\, \Lambda^{d/2} |\Omega|_h, \tag{1.10}$$

where $\mathbb{L}_{1,d}$ *is the constant from Theorem 1.1, so that*

$$(1 + d/2)\mathbb{L}_{1,d} \leq R_{1,1}(1 + d/2)\, \mathrm{L}^{\mathrm{cl}}_{1,d} = R_{1,1}\, \mathrm{L}^{\mathrm{cl}}_{0,d}.$$

The inequality (1.10) is a Pólya type inequality [26] for manifolds with constant negative curvature, where, we believe, the constant is not sharp.

Conjecture 1. *For the counting function $\mathcal{N}(\Lambda)$ of the eigenvalues*

$$\lambda_k = (d-1)^2/4 + \nu_k$$

of the Dirichlet boundary value problem (1.8) *we have*

$$\mathcal{N}(\Lambda) \leq \mathrm{L}^{\mathrm{cl}}_{0,d}\, \Lambda^{d/2}\, |\Omega|_h. \tag{1.11}$$

Remark 1.3. *At the moment we do not have any examples of Ω for which the inequality* (1.11) *holds.*

In Section 6 we consider the special case of Theorem 1.2, where Ω is a product domain $\Omega = \widetilde{\Omega} \times (a,b)$, and $\widetilde{\Omega} \subset \mathbb{R}^{d-1}$, is a domain of finite Lebesgue measure and $0 < a < b \leq \infty$. This additional structure of Ω allows us to obtain a better constant than the constant which could be derived from Theorem 1.1 in the case $1/2 \leq \gamma < 1$, see Theorem 6.1. Unfortunately it does not imply the improvement of the constant found in Theorem 1.2.

Finally in Section 7 we give an example of a domain in $\mathbb{H}^2$ supporting the conjecture 1 using numerics.

2. Some preliminary results

In [24] the author gives a simple proof of the fact that the continuous spectrum of the operator $-\Delta_h$ in dimension two coincides with the interval $[1/4,\infty)$ by using the Cauchy-Schwarz inequality. Besides, he gives a more complicated proof of the fact that in the case of $\mathbb{H}^d$, $d > 2$ the continuous spectrum fills the semi-axis $[(d-1)^2/4,\infty)$.

In this section we present a simple proof of following well-known fact.

Proposition 2.1. *Let $-\Delta_h$ be the Laplacian in $L^2\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$, $d \geq 2$. Then its spectrum is absolutely continuous and coincides with*

$$\sigma_c = [(d-1)^2/4, \infty).$$

Proof. Let us consider the quadratic form of the operator (1.1)

$$(-\Delta_h u, u) = \int_{\mathbb{H}^n} y^{2-d} \left(|\partial_y u|^2 + \sum_{n=1}^{d-1} |\partial_{x_n} u|^2 \right) dx\, dy.$$

The substitution

$$y = e^t, \qquad u = e^{\frac{d-1}{2}\,t}\, v, \tag{2.1}$$

implies

$$\iint_{\mathbb{H}^d} |u|^2 \frac{dx\, dy}{y^d} = \int_{\mathbb{R}^d} |v|^2\, dx\, dt.$$

and

$$(-\Delta_h u, u) = \int_{\mathbb{R}^d} \left(|\partial_t v|^2 + \frac{(d-1)^2}{4} |v|^2 + e^{2t} \sum_{n=1}^{d-1} |\partial_{x_n} v|^2 \right) dxdt.$$

Thus we reduce the hyperbolic Laplacian to the operator in $L^2(\mathbb{R}^d)$

$$-\frac{\partial^2}{\partial t^2} - e^{2t}\,\Delta_x + \frac{(d-1)^2}{4},$$

and (2.1) generates the isometry between $L^2\left(\mathbb{H}^d, \frac{dx\,dy}{y^d}\right)$ and $L^2\left(\mathbb{R}^d, dxdt\right)$. Obviously the spectrum of the differential part of the above expression coincides with $[0,\infty)$ and the term $\frac{(d-1)^2}{4}$ gives the required shift of the spectrum. The proof is complete. □

Remark 2.4. *It might be interesting to obtain a simple proof of properties of the spectrum of the operator $-\Delta_h$ in the case when the negative curvature is not a constant.*

In order to prove Theorem 1.1 we need to recall some results on 1D Schrödinger operators with operator-valued potentials.

Proposition 2.2. *Let $Q = Q(x) \geq 0$ be a self-adjoint operator-valued function in a Hilbert space G for almost every $x \in \mathbb{R}$. We assume that $\operatorname{Tr} Q(\cdot) \in L^{\gamma+1/2}(\mathbb{R}, G)$, $\gamma \geq 1/2$. Then*

$$\operatorname{Tr}\left(-\frac{d^2}{dx^2} \otimes I_G - Q\right)_-^{\gamma} \leq \mathrm{L}_{\gamma,1} \int_{\mathbb{R}} \operatorname{Tr} Q^{\gamma+1/2}\, dx,$$

where I_G is the identity operator in G, and $\mathrm{L}_{\gamma,1}$ is defined in (1.3).

If $\gamma = 1/2$ then the constant $\mathrm{L}_{\gamma=1/2,1} = 2\mathrm{L}^{\mathrm{cl}}_{1/2,1}$ and it is sharp. This was obtained in [8] and from this one immediately obtains that $\mathrm{L}_{\gamma,1} \leq 2\mathrm{L}^{\mathrm{cl}}_{\gamma,1}$ with $1/2 \leq \gamma < 1$. (For the scalar case the sharp constant in the case $\gamma = 1/2$ was obtained in [9]).

If $\gamma = 1$ then the sharp constant is unknown and the best known constant for some years was referred to [5] (see also [6]). It is only recently this constant was improved in the paper [10], where the authors have shown that $\mathrm{L}_{1,1} \leq R_{1,1}\mathrm{L}^{\mathrm{cl}}_{1,1}$ with $R_{1,1} \leq 1.456\ldots$. This leads to the improved estimate $\mathrm{L}_{\gamma,1} \leq R_{1,1}\mathrm{L}^{\mathrm{cl}}_{\gamma,1}$ with $1 \leq \gamma < 3/2$.

Finally for any $\gamma \geq 3/2$ the above Proposition was proved in [17] and in this case we have sharp constants $\mathrm{L}_{\gamma,1} = \mathrm{L}^{\mathrm{cl}}_{\gamma,1}$.

In all cases the authors first obtained inequalities for $\gamma = 1/2, 1$ and $3/2$ that were afterwards extended to arbitrary γ's using lifting argument found by Aizenman and Lieb [2].

3. The proof on the main result

Let us consider the quadratic form of the operator (1.1)

$$(-\Delta_h u, u) = \int_{\mathbb{H}^d} y^{2-d} \left(|\partial_y u|^2 + \sum_{n=1}^{d-1} |\partial_{x_n} u|^2 \right) dx\, dy.$$

Applying the exponential change of variables (2.1) we reduce the problem to the study of the spectrum defined by the form in $L^2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \left(|\partial_t v|^2 + e^{2t} \sum_{n=1}^{d-1} |\partial_{x_n} v|^2 - V(x, e^t)|v|^2 \right) dxdt = -\mu \int_{\mathbb{R}^d} |v|^2 \, dxdt.$$

Using the variational principle and the Lieb–Thirring inequalities for 1D Schrödinger operators with operator-valued symbols, see Proposition 2.2, we obtain

$$\begin{aligned} \sum \mu_k^\gamma(-\Delta_h - V) &\leq \sum \mu_k^\gamma \left(-\frac{d^2}{dt^2} - \left(e^{2t} \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2} + V(x, e^t) \right)_+ \right) \\ &\leq \mathrm{L}_{\gamma,1} \int_{\mathbb{R}} \mathrm{Tr} \left(-e^{2t} \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2} - V(x, e^t) \right)_-^{\gamma+1/2} dt. \end{aligned}$$

This trick reduced the problem to a Schrödinger operator in $L^2(\mathbb{R}^{d-1})$, where the exponent e^{2t} is just a parameter:

$$\begin{aligned} \mathrm{Tr} &\left(-e^{2t} \sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2} - V(x, e^t) \right)_-^{\gamma+1/2} \\ &= e^{2t(\gamma+1/2)} \mathrm{Tr} \left(-\sum_{n=1}^{d-1} \frac{\partial^2}{\partial x_n^2} - e^{-2t} V(x, e^t) \right)_-^{\gamma+1/2}. \end{aligned}$$

Applying in dimension $d-1$ the standard Lieb–Thirring inequality we find

$$\begin{aligned} \sum \mu_k^\gamma &\leq \mathrm{L}_{\gamma,1} \mathrm{L}_{\gamma+1/2,d-1} \int_{\mathbb{R}^d} e^{(1-d)t} V(x, e^t)^{\gamma+d/2} \, dxdt \\ &= \mathrm{L}_{\gamma,1} \mathrm{L}_{\gamma+1/2,d-1} \int_{\mathbb{H}^d} V(x, y)^{\gamma+d/2} \frac{dxdy}{y^d}. \end{aligned}$$

Therefore the constant $\mathbb{L}_{\gamma,d}$ in (1.6) satisfies

$$\mathbb{L}_{\gamma,d} \leq \mathrm{L}_{\gamma,1} \mathrm{L}_{\gamma+1/2,d-1},$$

and consequently satisfies the explicit bound (1.7) if we recall (1.2), (1.3) and the relation

$$\mathrm{L}^{\mathrm{cl}}_{\gamma,1}\mathrm{L}^{\mathrm{cl}}_{\gamma+1/2,d-1}=\mathrm{L}^{\mathrm{cl}}_{\gamma,d}.$$

The proof is complete.

4. Dual inequalities

Consider an orthonormal set of function $\{u_m\}_{m=1}^M$ in $L^2(\mathbb{H}^d, y^{-d}\,dxdy)$, which belong to the Sobolev space $H^1(\mathbb{H}^d, y^{-d}\,dxdy)$ with norm

$$\|u\|^2_{H^1}=\int_{\mathbb{H}^d} y^{-d}|u|^2\,dx\,dy+\int_{\mathbb{H}^d} y^{2-d}\left(|\partial_y u|^2+\sum_{n=1}^{d-1}|\partial_{x_n}u|^2\right)\,dx\,dy.$$

Using the exponential change of variables

$$y=e^t,\qquad u_m=e^{(d-1)t/2}\,v_m$$

we find

$$\delta_{m,l}=\int_{\mathbb{H}^d} u_m\overline{u}_l\frac{dxdy}{y^d}=\int_{\mathbb{R}^d} v_m\overline{v}_l\,dxdt.$$

Namely, this shows that if the functions $\{u_m\}_{m=1}^M$ are orthonormal in $L^2(\mathbb{H}^d, y^{-d}\,dxdy)$ then the functions $\{v_m\}_{m=1}^M$ are orthonormal in $L^2(\mathbb{R}^d)$.

Setting $\gamma=1$ we obtain by the variational principle for sums of eigenvalues (see [11, Corollary 1.35])

$$\sum_{m=1}^M\int_{\mathbb{R}^d}\left(|\partial_t v_m|^2+e^{2t}\sum_{n=1}^{d-1}|\partial_{x_n}v_m|^2-V(x,e^t)|v_m|^2\right)\,dxdt$$
$$\geq-\sum_m\mu_m\geq-\mathrm{L}_{1,d}\int_{\mathbb{R}^d}V(x,e^t)^{1+d/2}\,dxdt.$$

Thus,

$$\sum_{m=1}^M\int_{\mathbb{R}^d}\left(|\partial_t v_m|^2+e^{2t}\sum_{n=1}^{d-1}|\partial_{x_n}v_m|^2\right)\,dxdt$$
$$\geq\int_{\mathbb{R}^d}\left(V(x,e^t)\sum_{m=1}^M|v_m|^2-\mathrm{L}_{1,d}\,V(x,e^t)^{1+d/2}\right)\,dxdt$$
$$=\int_{\mathbb{R}^d}\left(V(x,e^t)\,\widetilde{\rho}-\mathrm{L}_{1,d}\,V(x,e^t)^{1+d/2}\right)\,dxdt,$$

where $\widetilde{\rho}=\sum_{m=1}^N|v_m|^2$. We now choose

$$V=\left(\frac{\widetilde{\rho}}{\mathrm{L}_{1,d}(1+d/2)}\right)^{2/d}$$

and find

$$\sum_{m=1}^{M}\int_{\mathbb{R}^d}\left(|\partial_t v_m|^2+e^{2t}\sum_{n=1}^{d-1}|\partial_{x_n}v_m|^2\right)dxdt \ge \mathrm{K}_{1,d}\int_{\mathbb{R}^d}(\widetilde{\rho})^{1+2/d}\,dxdt, \tag{4.1}$$

where

$$\mathrm{K}_{1,d}=\left[\frac{2}{d}\left(1+\frac{d}{2}\right)^{1+2/d}\mathrm{L}_{1,d}^{2/d}\right]^{-1}.$$

Returning to the orthonormal system of functions $\{u_m\}$ and denoting by $\rho=\sum_{m=1}^{M}|u_m|^2$ we obtain

$$\int_{\mathbb{R}^d}(\widetilde{\rho})^{1+2/d}\,dxdt=\int_{\mathbb{H}^d}y^{\frac{2(1-d)}{d}}\rho^{1+2/d}\,\frac{dxdy}{y^d}.$$

Besides, when passing from the quadratic forms on the left hand side of (4.1) to the quadratic forms $(-\Delta_h u_m,u_m)$ we have to add the shift

$$(d-1)^2/4\|u_m\|_{L^2(\mathbb{H}^d,y^{-d}dxdy)}=(d-1)^2/4,\quad m=1,2,\dots,M.$$

Finally we have

Theorem 4.1. *Let $\{u_m\}_{m=1}^{M}\in H^1(\mathbb{H}^d,y^{-d}dxdy)$ be an orthonormal system of function in $L^2(\mathbb{H}^d,y^{-d}dxdy)$. Then*

$$\sum_{m=1}^{M}\int_{\mathbb{H}^d}y^{2-d}\left(|\partial_y u_m|^2+\sum_{n=1}^{d-1}|\partial_{x_n}u_m|^2\right)dx\,dy$$
$$\ge \mathrm{K}_{1,d}\int_{\mathbb{H}^d}y^{\frac{2(1-d)}{d}}\rho^{1+2/d}\,\frac{dxdy}{y^d}+M\,\frac{(d-1)^2}{4}\,.$$

We now single out the one-function case (that is, $M=1$) and for a $u\in H^1(\mathbb{H}^d,y^{-d}dxdy)$ we denote

$$\|u\|=\left(\int_{\mathbb{H}^d}|u|^2\,\frac{dxdy}{y^d}\right)^{1/2}.$$

Then

Corollary 4.1. *For a function $u\in H^1(\mathbb{H}^d,y^{-d}dxdy)$ we have*

$$\|u\|^{4/d}\int_{\mathbb{H}^d}y^{2-d}\left(|\partial_y u|^2+|\nabla_x u|^2\right)dxdy$$
$$\ge \mathrm{K}_{1,d}\int_{\mathbb{H}^d}y^{\frac{2(1-d)}{d}}|u|^{2+4/d}\,\frac{dxdy}{y^d}+\frac{(d-1)^2}{4}\,\|u\|^{2+4/d}.$$

5. Proof of Theorem 1.2

Let $\Omega \subset \overline{\Omega} \subset \mathbb{H}^d$ be a domain of finite hyperbolic measure $|\Omega|_h < \infty$. Let us consider (in the sense of quadratic forms) the Dirichlet boundary value problem in $L^2(\Omega, y^{-d}dxdy)$, (see (1.8),(1.9))

$$\left(-\Delta_h - \frac{(d-1)^2}{4}\right) u = \nu u, \quad u|_{\partial\Omega} = 0.$$

The spectrum of this operator is discrete. In order to estimate such spectrum we introduce the Schrödinger operator in $L^2(\mathbb{H}^d, y^{-d}dxdy)$ with the potential

$$V = \begin{cases} \Lambda, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

Due to the variational principle (more precisely, by comparing the domains of the Dirichlet Laplacian and the Schrödinger operator and using the extension by zero) we see that the negative eigenvalues $-\mu_k$ of the operator

$$-\Delta_h - \frac{(d-1)^2}{4} - V$$

satisfy the inequality $\Lambda - \mu_k \leq \nu_k$. Therefore by applying Theorem 1.1 with $\gamma = 1$ we find

$$\sum_k (\Lambda - \nu_k)_+ \leq \sum_k \mu_k \leq \mathbb{L}_{1,d}\, \Lambda^{1+d/2} \int_\Omega \frac{dxdy}{y^d} = \mathbb{L}_{1,d}\, \Lambda^{1+d/2}\, |\Omega|_h.$$

Then for any $\Upsilon > \Lambda$

$$\mathcal{N}(\Lambda) \leq \frac{1}{\Upsilon - \Lambda} \sum_k (\Upsilon - \nu_k)_+ \leq \mathbb{L}_{1,d} \frac{\Upsilon^{1+d/2}}{\Upsilon - \Lambda} |\Omega|_h.$$

Minimising the right hand side of the latter inequality we find

$$\Upsilon = \Lambda \frac{1 + d/2}{d/2}$$

and recalling that $\mathbb{L}_{1,d} \leq \mathrm{L}_{1,d} \leq R_{1,1} \mathrm{L}^{\mathrm{cl}}_{1,d}$ we finally obtain

$$\mathcal{N}(\Lambda) \leq \left(1 + \frac{d}{2}\right) \left(1 + \frac{2}{d}\right)^{d/2} R_{1,1}\, \mathrm{L}^{\mathrm{cl}}_{1,d}\, |\Omega|_h\, \Lambda^{d/2}. \tag{5.1}$$

This concludes the proof of Theorem 1.2.

6. A SPECIAL CASE OF THEOREM 1.2

Let $\Omega = \widetilde{\Omega} \times (a, b) \subset \mathbb{H}^d$, where $\widetilde{\Omega} \subset \mathbb{R}^{d-1}$ and $0 < a < b \leq \infty$. We assume that the Lebesgue measure $|\widetilde{\Omega}| < \infty$, so that the hyperbolic measure is finite

$$|\Omega|_h = |\widetilde{\Omega}| \int_a^b \frac{dy}{y^d} < \infty.$$

We consider the Dirichlet problem in $L^2\left(\Omega, y^{-d}dxdy\right)$

$$-\Delta_h u = \lambda u, \qquad u\big|_{\partial\Omega} = 0,$$

Using the substitution (2.1) we reduce the problem to

$$\left(-\partial_t^2 - e^{2t}\Delta_x + \frac{(d-1)^2}{4}\right) v(x,t) = \lambda v(x,t) \qquad v\big|_{\partial(\widetilde{\Omega}\times(\alpha,\beta))} = 0, \tag{6.1}$$

where $\alpha = \ln a$, $\beta = \ln b$. As before it is convenient to introduce values ν

$$\lambda = \frac{(d-1)^2}{4} + \nu$$

Due to the product structure of Ω the eigenfunctions $\{v_{\ell k}\}_{\ell,k=1}^{\infty}$ of the problem can be found as the product

$$v_{\ell k}(x,t) = \varphi_\ell(x)\psi_{\ell k}(t),$$

where φ_ℓ satisfy the Dirichlet boundary value problem

$$-\Delta_x \varphi_\ell = \varkappa_\ell\, \varphi_\ell, \qquad \varphi_\ell\big|_{\partial\widetilde{\Omega}} = 0.$$

and

$$-\partial_t^2\, \psi_{\ell k}(t) + e^{2t}\, \varkappa_\ell\, \psi_{\ell k}(t) = \nu_{\ell k}\, \psi_{\ell k}(t), \qquad \psi_{\ell k}(t)\big|_{t=\alpha,\beta} = 0. \tag{6.2}$$

Note that the functions $\{\varphi_\ell\}_{\ell=1}^{\infty}$ give us an orthonormal basis in $L^2(\widetilde{\Omega})$ and for each fixed $\ell \in \mathbb{N}$ we have $\varkappa_\ell > 0$. Therefore the spectrum of operator (6.2) is discrete (including the case when $\beta = \infty$) and for each ℓ the set of functions $\{\psi_{\ell k}\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\alpha, \beta)$.

Altogether we have the equation

$$\left(-\partial_t^2 - e^{2t}\Delta_x\right)\, \varphi_\ell(x)\, \psi_{\ell k}(t) = \nu_{\ell k}\, \varphi_\ell(x)\, \psi_{\ell k}(t),$$

where

$$\left\{\frac{(d-1)^2}{4} + \nu_{\ell k}\right\}_{\ell,k=1}^{\infty}$$

are all eigenvalues of the problem (6.1).

Using the notations from Section 5 and applying Lieb–Thirring inequalities in Proposition 2.2 for the $1/2$-moment of the Schrödinger operator with the operator-valued potential $-e^{2t}\Delta_x - \Lambda$ with the sharp constant $2\,\mathrm{L}^{\mathrm{cl}}_{1/2,1}$ (see [8]) we obtain

$$\sum_{\ell,k=1}^{\infty} (\Lambda - \nu_{\ell k})_+^{1/2} = \mathrm{Tr}\,\left(-\partial_t^2 - e^{2t}\,\Delta_x - \Lambda\right)_-^{1/2}$$
$$\leq 2\,\mathrm{L}^{\mathrm{cl}}_{1/2,1} \int_\alpha^\beta \mathrm{Tr}\,\left(\Lambda + e^{2t}\Delta_x\right)_+ dt.$$

The multiplier e^{2t} in the study of the trace $\mathrm{Tr}\,(\Lambda + e^{2t}\Delta_x)_+$ could be considered as a constant. Therefore applying Berezin–Li &Yau inequality for the Dirichlet Laplacian in $\widetilde{\Omega} \subset \mathbb{R}^{d-1}$ (see [3],[21], [16] and also [11, Section 3.5]) we find

$$\mathrm{Tr}\,\left(\Lambda + e^{2t}\Delta_x\right)_+ = \sum_\ell (\Lambda - e^{2t}\varkappa_\ell)_+$$
$$\leq |\widetilde{\Omega}|\,(2\pi)^{1-d} \int_{\mathbb{R}^{d-1}} (\Lambda - e^{2t}\,|\xi|^2)_+\,d\xi = \mathrm{L}^{\mathrm{cl}}_{1,d-1}\,|\widetilde{\Omega}|\,\Lambda^{(d+1)/2}\,e^{(1-d)t}.$$

Finally by using

$$\mathrm{L}^{\mathrm{cl}}_{1/2,1}\mathrm{L}^{\mathrm{cl}}_{1,d-1} = \mathrm{L}^{\mathrm{cl}}_{1/2,d}$$

and returning to variables (y, x) we arrive at

$$\sum_{\ell,k=1}^{\infty} (\Lambda - \nu_{\ell k})_+^{1/2} \leq \Lambda^{\frac{d+1}{2}}\,2\,\mathrm{L}^{\mathrm{cl}}_{1/2,d} \int_\alpha^\beta e^{(1-d)t}\,dt\,|\widetilde{\Omega}|$$
$$= \Lambda^{\frac{d+1}{2}}\,2\,\mathrm{L}^{\mathrm{cl}}_{1/2,d} \int_\Omega \frac{dxdy}{y^d} = \Lambda^{\frac{d+1}{2}}\,2\,\mathrm{L}^{\mathrm{cl}}_{1/2,d}\,|\Omega|_h.$$

Using the standard Aizenman–Lieb arguments we can extent the above inequality to the γ-Riesz means with $1/2 \leq \gamma < 1$ and obtain

Theorem 6.1. *Let* $\Omega = \widetilde{\Omega} \times (a, b) \subset \mathbb{H}^d$*, where* $\widetilde{\Omega} \subset \mathbb{R}^{d-1}$ *with* $|\widetilde{\Omega}| < \infty$*, and further let* $0 < a < b \leq \infty$*. Then for the values* $\{\nu_{\ell k}\}_{\ell,k=1}^\infty$ *related to the eigenvalues of the Dirichlet Laplacian* (1.1) *via the equation*

$$\lambda_{\ell k} = \frac{(d-1)^2}{4} + \nu_{\ell k}$$

we have

$$\sum_{\ell,k=1}^{\infty} (\Lambda - \nu_{\ell k})_+^\gamma \leq \Lambda^{\frac{d}{2}+\gamma}\,2\,\mathrm{L}^{\mathrm{cl}}_{\gamma,d}\,|\Omega|_h, \quad 1/2 \leq \gamma < 1. \tag{6.3}$$

Remark 6.5. *The constant* $2\mathrm{L}^{\mathrm{cl}}_{\gamma,d}$*,* $1/2 \leq \gamma < 1$ *is better than the constant* $2\,R_{1,1}\,\mathrm{L}^{\mathrm{cl}}_{\gamma,d}$ *that could be obtained from Theorem 1.1.*

Similarly to Section 5 we can use the inequality (6.3) for estimating the counting function $\mathcal{N}(\Lambda)$ for spectrum of the Dirichlet Laplacian (1.1) in domain with the product structure. Indeed, for any $\Upsilon > \Lambda$

$$\mathcal{N}(\Lambda) \le \frac{1}{(\Upsilon - \Lambda)^{1/2}} \sum_{\ell,k=1}^{\infty} (\Upsilon - \nu_{\ell k})_+^{1/2} \le 2\mathrm{L}^{\mathrm{cl}}_{1/2,d} \frac{\Upsilon^{(d+1)/2}}{(\Upsilon - \Lambda)^{1/2}} \, |\Omega|_h .$$

Minimising the right hand side of the latter inequality we find $\Upsilon = \Lambda \frac{1+d}{d}$ and thus

$$\mathcal{N}(\Lambda) \le \left(\frac{d+1}{d} \right)^{(d+1)/2} \sqrt{d}\, 2\mathrm{L}^{\mathrm{cl}}_{1/2,d} \, |\Omega|_h \, \Lambda^{d/2}. \tag{6.4}$$

However, the ratio of the constants (6.4) and (5.1) is greater than one and therefore Theorem 6.1 does not imply any improvement for the inequality (5.1).

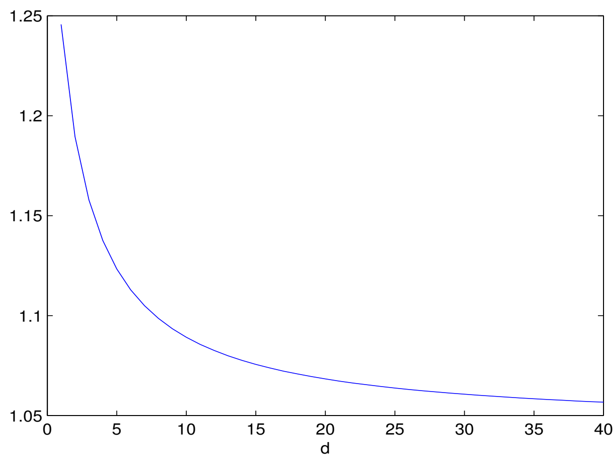

FIGURE 1. The graph of the ratio of the constants on the right-hand sides in (6.4) and (5.1), respectively.

7. A NUMERICAL EXAMPLE SUPPORTING CONJECTURE 1

Let $d = 2$ and let

$$\Omega = (0, \pi) \times (e^{-1}, e) = \big\{ (x, y) : \, x \in (0, \pi), \, y \in (e^{-1}, e) \big\}$$

with

$$|\Omega|_h = \int_0^\pi dx \int_{e^{-1}}^{e} \frac{dy}{y^2} = \pi (e - e^{-1}).$$

Consider the Dirichlet Laplacian $-\Delta_h$ in $L^2(\Omega, y^{-2}dxdy)$ defined in (1.1). Using the notations from Section 6 we have $\alpha = -1$, $\beta = 1$. Obviously the eigenvalues of the problem

$$-\partial_x^2 \varphi = \varkappa \varphi, \quad \varphi\big|_{x=0,\pi} = 0$$

are $\varkappa_\ell = \ell^2$, $\ell \in \mathbb{N}$.

The arguments from Section 6 imply that the problem is reduced to the study of the eigenvalues $\nu_{\ell k}$ satisfying the equation

$$-\partial_t^2 \, \psi_{\ell k}(t) + e^{2t} \, \ell^2 \, \psi_{\ell k}(t) = \nu_{\ell k} \, \psi_{\ell k}(t), \qquad \psi_{\ell k}(t)\big|_{t=-1,1} = 0. \tag{7.1}$$

Now comes numerics to prove the following Pólya type inequality (see Fig. 2):

$$\mathcal{N}(\Lambda) = \#\{\ell, k : \, \nu_{\ell k} < \Lambda\} \le (2\pi)^{-2} \int_0^\pi \int_{-1}^1 \int_{\xi_2^2 + e^{2t}\xi_1^2 < \Lambda} d\xi_1 d\xi_2 dt dx$$

$$= (2\pi)^{-2} \pi \Lambda \int_{-1}^1 e^{-t} dt \int_{\xi_1^2 + \xi_2^2 < 1} d\xi_2 d\xi_1 = \frac{1}{4}\Lambda \left(e - e^{-1}\right) = \mathrm{L}_{0,2}^{\mathrm{cl}} \Lambda |\Omega|_h.$$

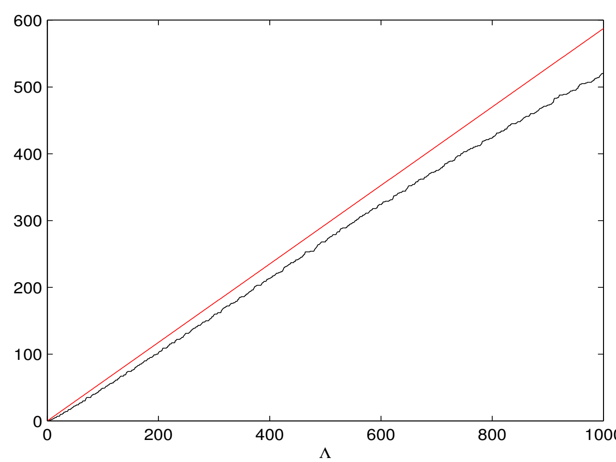

FIGURE 2. The graph $\mathcal{N}(\Lambda)$ shown in black and the graph of $L_{0,2}^{\mathrm{cl}} \Lambda |\Omega|_h$ is shown in red.

Let us say a few words about the calculations of the eigenvalues of the collection of problems (7.1). We use the Chebyshev differentiation matrix [28] for the spectral approximation of the derivative (and this matrix squared for the second derivative) for the numerical solution of the eigenvalue problems (7.1) (we observe that the potentials are analytic). We have used matrices of order 400×400. The accuracy is tested against the problem (7.1) with $l = 0$, so that the corresponding eigenvalues $n^2 \cdot \frac{4}{\pi^2}$ are computed for $n = 1, \dots, 200$ with correct 14 decimal places. We therefore reasonably expect that the accuracy is of the similar order for $\ell \geq 1$.

We set $\Lambda \leq 1000$. Then to calculate $\mathcal{N}(\Lambda)$ it is enough to limit $l \leq 50$, since the first eigenvalue of (7.1) with $l = 50$ is already greater than 1000. The eigenvalues $\nu_{\ell k}$ are of the order $k^2 \cdot \frac{4}{\pi^2}$ and therefore the length $[1 : 200]$ of each the sequence of eigenvalues $\nu_{\ell k}$ taken into account for each fixed $\ell \leq 50$ is also more than enough for $\Lambda \leq 1000$.

Acknowledgements: The authors would like to thank N. A. Zaitsev for his help and advice concerning the computer calculations of the eigenvalues of the Sturm–Liouville problems (7.1). We also thank N. D. Filonov and the anonymous referees for their remarks and comments that have helped to improve the paper.

The work of A.I. was supported by Moscow Center for Fundamental and Applied Mathematics, Agreement with the Ministry of Science and Higher Education of the Russian Federation, No. 075-15-2022-283.

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