Hardy inequalities for simply connected planar domains

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To Mikhail Shlëmovich Birman on his 80-th birthday

ABSTRACT. In 1986 A. Ancona showed, using the Koebe one-quarter Theorem, that for a simply-connected planar domain the constant in the Hardy inequality with the distance to the boundary is greater than or equal to $1/16$. In this paper we consider classes of domains for which there is a stronger version of the Koebe Theorem. This implies better estimates for the constant appearing in the Hardy inequality.

1. Main result and discussion

Let Ω be a domain in $\mathbb{R}^2$ and let $\Omega^c = \mathbb{R}^2 \setminus \Omega$ be its complement. It is known that there exists a number $r = r(\Omega) > 0$ such that for any function $u \in \mathsf{C}_0^1(\Omega)$ we have:

$$(1.1) \qquad \int_\Omega |\nabla u|^2 d\mathbf{x} \geq r^2 \int_\Omega \frac{|u|^2}{\delta(\mathbf{x})^2} d\mathbf{x}, \quad \delta(\mathbf{x}) = \inf_{\mathbf{y} \in \Omega^c} |\mathbf{y} - \mathbf{x}|,$$

e.g. E.B. Davies [4], [5], [6] and V.G. Maz'ya [10]. For convex domains $r = 1/2$ and it is sharp, see e.g. [4]. However, the sharp constant for non-convex domains is unknown, although for arbitrary planar simply-connected domains A. Ancona [1] proved (1.1) with $r = 1/4$. Some specific examples of non-convex domains were considered in [6] (see also J. Tidblom [12]). For example, it was found that if $\Omega = \mathbb{R}^2 \setminus \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty)$, then $r^2 = 0.20538....$

Our objective is to obtain the Hardy inequality for simply-connected non-convex domains $\Omega \subset \mathbb{R}^2$, whose degree of non-convexity can be "quantified". We introduce two possible "measures" of non-convexity.

Let $\Lambda \subset \mathbb{C}$ be a simply-connected domain such that $0 \subset \partial\Lambda$. Denote by $\Lambda(w, \phi) = e^{i\phi}\Lambda + w$ the transformation of Λ by rotation by angle $\phi \in (-\pi, \pi]$ in the positive direction and translation by $w \in \mathbb{C}$:

$$(1.2) \qquad \Lambda(w, \phi) = \{z \in \mathbb{C} : e^{-i\phi}(z - w) \in \Lambda\}.$$

Denote by $K_\theta \subset \mathbb{C}$, $\theta \in [0, \pi]$ the sector

$$(1.3) \qquad K_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}.$$

In words, this is an open sector symmetric with respect to the real axis, with the angle 2θ at the vertex. Here and below we always assume that $\arg \zeta \in (-\pi, \pi]$ for all $\zeta \in \mathbb{C}$. Our first assumption on the domain Ω is the following

CONDITION 1.1. *There exists a number $\theta \in [0, \pi]$ such that for each $w \in \Omega^c$ one can find a $\phi = \phi_w \in (-\pi, \pi]$ such that*

$$\Omega \subset K_\theta(w, \phi_w).$$

Very loosely speaking, this means that the domain Ω satisfies the exterior cone condition. The difference is of course that the cone is now supposed to be infinite. Because of this, Condition 1.1 is equivalent to itself if stated for the boundary points $w \in \partial\Omega$ only.

Note also that if Condition 1.1 is satisfied for some θ, then automatically $\theta \geq \pi/2$, and the equality $\theta = \pi/2$ holds for convex domains.

THEOREM 1.2. *Suppose that the domain $\Omega \subset \mathbb{R}^2, \Omega \neq \mathbb{R}^2$ satisfies Condition 1.1 with some $\theta \in [\pi/2, \pi]$. Then for any $u \in \mathsf{C}_0^1(\Omega)$ the Hardy inequality (1.1) holds with*

$$(1.4) \qquad r = \frac{\pi}{4\theta}.$$

It is clear that the constant r runs from $1/4$ to $1/2$ when θ varies from π to $\pi/2$. For the domain $\Omega = K_\theta$ Theorem 1.2 does not give the best known result, found in [6], saying that the value of r remains equal to $1/2$ for the range $\theta \in [0, \theta_0]$ where $\theta_0 \approx 2.428$, which is considerably greater than $\pi/2$.

To describe another way to characterize the non-convexity, for $a > 0$ and $\theta \in [0, \pi)$, introduce the domains

$$(1.5) \qquad \tilde{D}_a = \{z \in \mathbb{C} : |z| > a \ \& \ |\arg z| \neq \pi\}, \ \ D_{a,\theta} = \tilde{D}_a(-ae^{i\theta}, 0).$$

The domain $\tilde{D}_a$ is the exterior of the disk of radius a centered at the origin with an infinite cut along the negative real semi-axis.

CONDITION 1.3. *There exist numbers $a > 0$ and $\theta_0 \in [0, \pi)$ such that for any $w \in \partial\Omega$ one can find a $\phi = \phi_w \in (-\pi, \pi]$ and $\theta \in [-\theta_0, \theta_0]$ such that*

$$\Omega \subset D_{a,\theta}(w, \phi_w).$$

Note that any domain satisfying Conditions 1.1 or 1.3, is automatically simply-connected.

The following Theorem applies to the domains with a finite in-radius

$$\delta_{\mathrm{in}} = \sup_{z \in \Omega} \delta(z).$$

THEOREM 1.4. *Let $\Omega \subset \mathbb{R}^2$, $\Omega \neq \mathbb{R}^2$ be a domain such that $\delta_{\mathrm{in}} < \infty$. Suppose that Ω satisfies Condition 1.3 with some $\theta_0 \in [0, \pi)$ and that*

$$(1.6) \qquad 2\delta_{\mathrm{in}} \leq R_0(a), \quad R_0(a) = \frac{a}{2(2^{1/2}|\tan(\theta_0/2)| + 1)}.$$

Then the Hardy inequality (1.1) holds with

$$(1.7) \qquad r = \frac{1}{2}\left[1 - \frac{2\delta_{\mathrm{in}}}{R_0(a)}\right].$$

A natural example of a domain to apply the above theorem, is the following horseshoe-shaped domain

$$\Lambda = \{z \in \mathbb{C} : \rho < |z| < \rho + \delta, \ |\arg z| < \psi, \ \operatorname{Re} z > \rho \cos \psi\}, \quad \psi \in (0, \pi),$$

with $\rho, \delta > 0$. Simple geometric considerations show that this domain satisfies Condition 1.3 with $a = \rho$ and

$$\theta_0 = \begin{cases} 0, & \psi \leq \pi/2, \\ 2\psi - \pi, & \psi > \pi/2. \end{cases}$$

Assuming that $\delta\rho^{-1}$ is small, so that $\delta_{\mathrm{in}} = \delta$, we deduce from Theorem 1.4 that the Hardy inequality holds with a constant r, which gets close to $1/2$ as $\delta\rho^{-1} \to 0$. On the other hand, if $\delta_{\mathrm{in}}\rho^{-1}$ is large, one could apply Theorem 1.2, noticing that Λ satisfies Condition 1.1 with $\theta = (\pi + \psi)/2$, which gives the Hardy inequality with constant

$$r = \frac{1}{2(1 + \psi\pi^{-1})}.$$

which is obviously independent of δ_{in} or ρ.

Let us mention briefly some other recent results for convex domains, concerning the Hardy inequality with a remainder term. In the paper [3] H.Brezis and M. Marcus showed that if $\Omega \in \mathbb{R}^d$, $d \geq 2$, then the inequality could be improved to include the L^2-norm:

$$(1.8) \qquad \int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{\delta(\mathbf{x})^2} + C(\Omega) \int_\Omega |u|^2 d\mathbf{x},$$

where the constant $C(\Omega) > 0$ depends on the diameter of Ω. They also conjectured that $C(\Omega)$ should depend on the Lebesgue measure of Ω. This conjecture was justified in [9] and later generalised to L^p-type inequalities in [11]. Later S. Filippas, V.G. Maz'ya and A. Tertikas [8] (see also F.G. Avkhadiev [1]) obtained for $C(\Omega)$ an estimate in terms of the in-radius δ_{in}.

2. A version of the Koebe Theorem

A. Ancona has pointed out in [1] (page 278) that the Hardy inequality for simply-connected planar domains can be obtained from the famous Koebe one-quarter Theorem. Let f be a conformal mapping (i.e. analytic univalent) defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the condition $f(0) = 0$, $f'(0) = 1$. Denote by Ω the image of the disk under the function f, i.e. $\Omega = f(\mathbb{D})$, and set

$$\delta(\zeta) = \operatorname{dist}\{\zeta, \partial\Omega\} = \inf_{w \notin \Omega} |w - \zeta|$$

to be the distance from the point $\zeta \in \Omega$ to the boundary $\partial\Omega$. The classical Koebe one-quarter Theorem tells us that

$$\delta(0) \geq r,$$

with $r = 1/4$. On the other hand, if the domain Ω is convex, then it is known that $r = 1/2$, see e.g. P.L.Duren [7], Theorem 2.15. Without the normalization conditions $f(0) = 0$, $f'(0) = 1$ the above estimate can be rewritten as follows:

$$(2.1) \qquad \delta(f(0)) \geq r|f'(0)|.$$

For any simply-connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ we denote by $\mathbb{A}(\Omega)$ the class of all conformal maps such that $f(\mathbb{D}) = \Omega$.

Our proof of the main Theorems 1.2 and 1.4 relies on a version of the Koebe theorem, in which the constant r assumes values in the interval $[1/4, 1/2]$. We begin with a general statement which deduces the required Koebe-type result by comparing the domain Ω with some suitable "reference" domain. Let $\Lambda \subset \mathbb{C}, \Lambda \neq \mathbb{C}$ be a simply-connected domain such that $0 \subset \partial\Lambda$, and let g be a conformal function which maps Λ onto the complex plane with a cut along the negative semi-axis, i.e. onto

$$\Pi = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\},$$

such that $g(0) = 0$. We call Λ a *standard domain* and g - a *conformal map associated with the standard domain* Λ.

LEMMA 2.1. *Let $w \in \partial\Omega$ and suppose that for some standard domain Λ the inclusion*

$$(2.2) \qquad\qquad\qquad \Omega \subset \Lambda(w, \phi)$$

holds with some $\phi \in (-\pi, \pi]$. Let g be a conformal map associated with Λ,

(i) *Suppose that there are numbers $M > 0$ and $R_0 > 0$ such that for all $R \in (0, R_0]$*

$$(2.3) \qquad\qquad\qquad \left|\frac{g'(z)}{g(z)}\right| \geq \frac{\beta}{|z|}$$

for $z : 0 < |z| \leq R, z \in \Lambda$ with some $\beta = \beta(R) \in (0, M]$. Then for any $f \in \mathbb{A}(\Omega)$ satisfying the condition $M|f'(0)| \leq 4R_0$, the inequality

$$(2.4) \qquad\qquad |f(0) - w| \geq \frac{\beta(R_1)}{4}|f'(0)|, \quad R_1 = \frac{M|f'(0)|}{4},$$

holds.

(ii) *Suppose that the bound (2.3) holds for all $z \in \Lambda$ with some positive function $\beta = \beta(z)$. Then the inequality*

$$(2.5) \qquad\qquad |f(0) - w| \geq \frac{1}{4} \inf_{z \in \Lambda} \beta(z)|f'(0)|,$$

holds.

PROOF. Since $\Omega \subset \Lambda(w, \phi)$, the function

$$h(z) = g\big(e^{-i\phi}(f(z) - w)\big)$$

is conformal on $\mathbb{D}$, with values in Π. Since $0 \notin \Pi$, by the classical Koebe Theorem,

$$|h(0)| \geq \frac{1}{4}|h'(0)|,$$

so that

$$(2.6) \qquad |g\big(e^{-i\phi}(f(0) - w)\big)| \geq \frac{1}{4}|g'\big(e^{-i\phi}(f(0) - w)\big)||f'(0)|.$$

Suppose that the conditions of part (i) are fulfilled. If $|f(0) - w| \geq M|f'(0)|/4$, then there is nothing to prove, so we assume that $|f(0) - w| \leq M|f'(0)|/4$. Then by the assumption (2.3) we get

$$\frac{1}{\beta\big(M|f'(0)|/4\big)}|f(0) - w| \geq \frac{1}{4}|f'(0)|,$$

which leads to the required estimate (2.4).

If the conditions of part (ii) are fulfilled, then (2.5) follows directly from (2.6) and (2.3).

$\square$

COROLLARY 2.2. *Let $\Omega \subset \mathbb{C}$ be a domain. Suppose that for any $w \in \partial\Omega$ there is a standard domain $\Lambda = \Lambda_w$, such that the inclusion (2.2) holds with some $\phi = \phi_w \in (-\pi, \pi]$. Let $g = g_w$ be the conformal map associated with Λ_w.*

(i) *Suppose that g_w satisfies (2.3) for all $0 < |z| \leq R$, $z \in \Lambda_w$, for all $R \in (0, R_0]$ with some $\beta(R) = \beta_w(R) \in (0, M]$, where $M > 0$ and $R_0 > 0$ are independent of w. Then under the condition $M\delta_{\mathrm{in}} \leq R_0$ the estimate (2.1) holds for all $f \in \mathbb{A}(\Omega)$ with*

$$r = \frac{1}{4} \inf_{w \in \partial\Omega} \beta_w(R'), \ R' = M\delta_{\mathrm{in}}.$$

(ii) *Suppose that g_w satisfies (2.3) for all $z \in \Lambda_w$ with some $\beta(z) = \beta_w(z) > 0$. Then the estimate (2.1) holds for all $f \in \mathbb{A}(\Omega)$ with*

$$r = \frac{1}{4} \inf_{w \in \partial\Omega} \inf_{z \in \Lambda_w} \beta_w(z).$$

Observe that under the conditions of this corollary, the domain Ω is automatically simply-connected and $\Omega \neq \mathbb{C}$.

PROOF. (i) By the classical Koebe Theorem $|f'(0)| \leq 4\delta(f(0)) \leq 4\delta_{\mathrm{in}}$, so that under condition $M\delta_{\mathrm{in}} \leq R_0$ Lemma 2.1(i) entails the estimate (2.4) for each $w \in \partial\Omega$. Since $R_1 \leq R'$ and $\beta_w(\,\cdot\,)$ is a decreasing function, the required result follows.

Part (ii) immediately follows from Lemma 2.1(ii). $\square$

Now we apply the above results in the cases of standard domains K_θ and $D_{a,\theta}$, see (1.3) and (1.5) for definitions.

THEOREM 2.3. *Suppose that Ω satisfies Condition 1.1 with some $\theta \in [\pi/2, \pi]$. Then for any $f \in \mathbb{A}(\Omega)$ the inequality (2.1) holds with r given by (1.4).*

PROOF. Due to Condition 1.3, for each $w \in \partial\Omega$ we have $\Omega \subset K_\theta(w, \phi)$ with some $\phi \in (-\pi, \pi]$. Clearly, the domain K_θ is standard and the function

$$g(z) = z^\alpha, \ \alpha = \frac{\pi}{\theta}$$

is a conformal map associated with K_θ. One immediately obtains:

$$\frac{g'(z)}{g(z)} = \alpha \frac{z^{\alpha-1}}{z^\alpha} = \frac{\alpha}{z}, z \in \Lambda,$$

so that the conditions of Corollary 2.2(ii) hold with the constant $\beta = \alpha$. Now Corollary 2.2(ii) leads to the proclaimed result. $\square$

Note that for convex domains the angle θ is $\pi/2$, and hence we recover the known result $r = 1/2$. Actually, the proof of Lemma 2.1 is modelled on that for convex domains, which is featured in [**7**], Theorem 2.15.

Let us prove a similar result for Condition 1.3:

THEOREM 2.4. *Suppose that Ω satisfies Condition 1.3 with some $a > 0$, $\theta_0 \in [0, \pi)$, that $\delta_{\mathrm{in}} < \infty$, and that the condition (1.6) is fulfilled. Then for any $f \in \mathbb{A}(\Omega)$ the inequality (2.1) holds with r given by (1.7).*

PROOF. Due to Condition 1.3, for each $w \in \partial\Omega$ we have $\Omega \subset D_{a,\theta}(w, \phi)$ with some $\theta \in [0, \theta_0]$ and $\phi \in (-\pi, \pi]$. Clearly, the domain $D_{a,\theta}$ is standard and the function

$$g(z) = h^2(za^{-1}), \quad h(\zeta) = \sqrt{\zeta + e^{i\theta}} - \frac{1}{\sqrt{\zeta + e^{i\theta}}} - 2ib, \quad b = \sin\frac{\theta}{2},$$

is a conformal map associated with $D_{a,\theta}$. Write:

$$h(\zeta) = \frac{\zeta + e^{i\theta} - 1 - 2ib\sqrt{\zeta + e^{i\theta}}}{\sqrt{\zeta + e^{i\theta}}}$$

$$= \frac{\zeta + 2ibe^{i\theta/2} - 2ib\sqrt{\zeta + e^{i\theta}}}{\sqrt{\zeta + e^{i\theta}}}$$

$$= \frac{\psi(\zeta)}{\sqrt{\zeta + e^{i\theta}}}.$$

A direct calculation shows that

$$\frac{g'(z)}{g(z)} = \frac{2\psi'(z/a)}{a\psi(z/a)} - \frac{1}{z + ae^{i\theta}}.$$

Let us investigate the function ψ in more detail. Assume that $|\zeta| \le 1/2$. Rewrite:

$$\psi(\zeta) = \zeta + 2ibe^{i\theta/2} - 2ibe^{i\theta/2}\sqrt{1 + \zeta e^{-i\theta}}$$

$$= \zeta + 2ibe^{i\theta/2} - 2ibe^{i\theta/2}\left(1 + \frac{1}{2}\zeta e^{-i\theta} + \zeta\gamma_1\right)$$

$$= (1 - ibe^{-i\theta/2})\zeta - 2ibe^{i\theta/2}\zeta\gamma_1 = e^{-i\theta/2}\cos\frac{\theta}{2}\zeta - 2ibe^{i\theta/2}\zeta\gamma_1,$$

where

$$|\gamma_1(\zeta)| \le 2^{-3/2}|\zeta|, \quad |\zeta| \le 1/2.$$

Let's look at the derivative:

$$\psi'(\zeta) = 1 - \frac{ib}{\sqrt{\zeta + e^{i\theta}}}$$

$$= 1 - e^{-i\theta/2}\frac{ib}{\sqrt{1 + \zeta e^{-i\theta}}} = 1 - ibe^{-i\theta/2}(1 + \gamma_2)$$

$$= e^{-i\theta/2}\cos\frac{\theta}{2} - ibe^{-i\theta/2}\gamma_2,$$

where

$$|\gamma_2(\zeta)| \le 2^{1/2}|\zeta|, \quad |\zeta| \le 1/2.$$

Therefore

$$\frac{g'(z)}{g(z)} = \frac{2e^{-i\theta/2}\cos\frac{\theta}{2} - 2ibe^{-i\theta/2}\gamma_2(z/a)}{e^{-i\theta/2}\cos\frac{\theta}{2}z - 2ibe^{i\theta/2}z\gamma_1(z/a)} - \frac{1}{z + ae^{i\theta}}$$

$$= \frac{1}{z}\left[\frac{2 - 2i\tan(\theta/2)\gamma_2(z/a)}{1 - 2i\tan(\theta/2)e^{i\theta}\gamma_1(z/a)} - \frac{z}{a}\frac{1}{z/a + e^{i\theta}}\right]$$

Denote $\zeta = z/a$, $t = |\tan(\theta/2)|$. Using the above bounds on γ_1, γ_2, we obtain for $|\zeta| \leq 1/2$ the inequality

$$\left|\frac{g'(z)}{g(z)}\right| \geq \frac{2}{|z|}\left[\frac{1 - t|\gamma_2(\zeta)|}{1 + 2t|\gamma_1(\zeta)|} - |\zeta|\right]$$

$$\geq \frac{2}{|z|}\left[\frac{1 - 2^{1/2}t|\zeta|}{1 + 2^{-1/2}t|\zeta|} - |\zeta|\right].$$

Remembering the simple inequality

$$\frac{1-a}{1+b} \geq 1 - a - b,$$

satisfied for all non-negative a and b, from here we deduce that

$$\left|\frac{g'(z)}{g(z)}\right| \geq \frac{2}{|z|}\left[1 - \left(2^{3/2}|\tan(\theta/2)| + 1\right)\frac{|z|}{a}\right], \quad |z|/a \leq 1/2.$$

Therefore the condition (2.3) is satisfied for all $0 < |z| \leq R$, where

$$R \leq \min\left\{\frac{a}{2}, \; \frac{a}{2^{3/2}|\tan(\theta/2)| + 1}\right\}.$$

In any case, (2.3) is satisfied for

$$R \leq R_0(a, \theta) = \frac{a}{2^{3/2}|\tan(\theta/2)| + 2},$$

with

$$\beta(R) = \beta(R, \theta) = 2\left[1 - \frac{R}{R_0(a, \theta)}\right].$$

Note that $\beta \leq M$ with $M = 2$ and $\beta(R, \theta) \geq \beta(R, \theta_0)$, so that by Corollary 2.2(i) the estimate (2.1) holds with the r given by (1.7). $\qquad\square$

3. Proof of Theorems 1.2, 1.4

As soon as the Koebe Theorem (2.1) is established, our proof of the Hardy inequality follows that by A.Ancona [1]. Namely, our starting point is the inequality for the half-plane, which is an immediate consequence of the classical Hardy inequality in one dimension. Below we use the usual notation $z = x + iy$, $x, y \in \mathbb{R}$.

PROPOSITION 3.1. *Let* $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. *For any* $u \in \mathsf{C}_0^1(\mathbb{C}_+)$ *one has*

$$\int_{\mathbb{C}_+} |\nabla u|^2 dx dy \geq \frac{1}{4}\int_{\mathbb{C}_+} \frac{|u|^2}{x^2} dx dy.$$

Theorems 1.2 and 1.4 immediately follow from Proposition 3.1 with the help of the following conditional result:

THEOREM 3.2. *Let* $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ *be a simply connected domain. Suppose that for all* $g \in \mathbb{A}(\Omega)$ *the inequality* (2.1) *holds with some* $r \in [1/4, 1/2]$. *Then for any conformal mapping* $f : \mathbb{C}_+ \to \Omega$ *the following version of the Koebe Theorem holds:*

$$(3.1) \qquad\qquad \delta(f(z)) \geq 2rx|f'(z)|.$$

For $r = 1/4$ the estimate (3.1) can be found in [1]. For the reader's convenience we provide a proof of (3.1).

PROOF. For a conformal mapping $f : \mathbb{C}_+ \to \Omega$ and arbitrary $z \in \mathbb{C}_+$ we define

$$g(w) = g_z(w) = f\big(h(w)\big), \ h(w) = \frac{\overline{z}w + z}{1 - w},$$

where $w \in \mathbb{D}$. It is clear that for each fixed $z \in \mathbb{C}_+$ the function h maps $\mathbb{D}$ onto $\mathbb{C}_+$ and $h(0) = z, \ g(0) = f(z)$. The derivative is

$$g'(w) = \frac{z + \overline{z}}{(1 - w)^2} f'\big(h(w)\big),$$

so that

$$g'(0) = 2xf'(z).$$

Therefore, the Koebe theorem (2.1) implies that

$$\delta\big(f(z)\big) = \delta\big(g(0)\big) \geq r|g'(0)| = 2rx|f'(z)|,$$

as required. $\square$

PROOF OF THEOREMS 1.2, 1.4. According to Theorems 2.3 or 2.4 the Koebe Theorem for functions $f \in \mathbb{A}(\Omega)$ holds with the values of r given by (1.4) or (1.7) respectively.

Let $f : \mathbb{C}_+ \to \Omega$ be a conformal map. Remembering that conformal maps preserve the Dirichlet integral, from Proposition 3.1 we get for any $u \in \mathsf{C}_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 d\mathbf{x} = \int_{\mathbb{C}_+} |\nabla (u \circ f)|^2 dx dy \geq \frac{1}{4} \int_{\mathbb{C}_+} \frac{|(u \circ f)|^2}{x^2} d\mathbf{x}$$

$$= r^2 \int_{\mathbb{C}_+} \frac{|(u \circ f)|^2}{(2rx)^2 |f'(z)|^2} |f'(z)|^2 d\mathbf{x} \geq r^2 \int_\Omega \frac{|u|^2}{\delta(\mathbf{x})^2} d\mathbf{x}.$$

At the last step we have used (3.1).

Now Theorems 1.2, 1.4 follow. $\square$

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