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Spectral inequalities for Partial Differential Equations and their applications.

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1. Discrete Negative Spectrum of Schrödinger operators

1.1. Lieb-Thirring inequalities. Let us consider a self-adjoint Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta + V, \tag{1.1}$$

where V is a real-valued function. If the potential function V decays rapidly enough and, for example, smooth, then the spectrum of the operator H typically is absolutely continuous on $[0, \infty)$. If V has a non-trivial negative part, then H might have finite or infinite number of negative eigenvalues $\{\lambda_n(H)\}$. In case the number of negative eigenvalues is infinite, the point zero is the only possible accumulating point. The inequalities

$$\sum_n |\lambda_n| \le \frac{R_{\gamma,d}}{(2\pi)^d} \iint_{\mathbb{R}^2 d} (|\xi|^2 + V(x))_-^\gamma \, d\xi dx \le L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \, dx \tag{1.2}$$

are known as Lieb-Thirring bounds. Here and in the following, $V_\pm = (|V| \pm V)/2$ denote the positive and negative parts of the function V.

It is known that the inequality (1.2) holds true with some finite constants if and only if $\gamma \ge 1/2$, $d = 1$; $\gamma > 0$, $d = 2$ and $\gamma \ge 0$, $d \ge 3$. There are examples showing that (1.2) fails for $0 \le \gamma < 1/2$, $d = 1$ and $\gamma = 0$, $d = 2$.

Almost all the cases except for $\gamma = 1/2$, $d = 1$ and $\gamma = 0$, $d \ge 3$ were justified in the original paper of E.H.Lieb and W.Thirring [**LT**]. The critical case $\gamma = 0$, $d \ge 3$ is known as the Cwikel-Lieb-Rozenblum inequality, see [**Cw, L1, Roz1**] and also later proved in [**LY, Con**]. The remaining case $\gamma = 1/2$, $d = 1$ was verified by T.Weidl in [**W1**].

1.2. Weyl's asymptotics. Inequalities (1.2) play a crucial role in establishing Weyl's asymptotic formulae for the negative eigenvalues of the Schrödinger operator

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$-\Delta + \alpha V$ when the coupling constant $\alpha \to \infty$. Namely

$$(1.3)\qquad \lim_{\alpha\to+\infty} \alpha^{-\gamma-\frac{d}{2}} \sum_n |\lambda_n(\alpha)| = \lim_{\alpha\to+\infty} \alpha^{-\gamma-\frac{d}{2}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|\xi|^2 + \alpha V)_-^\gamma \frac{dx d\xi}{(2\pi)^d} = L^{cl}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\, dx\,,$$

where the so-called "classical constants" $L^{cl}_{\gamma,d}$ are defined by

$$(1.4)\qquad L^{\rm cl}_{\gamma,d} = (2\pi)^{-d} \int_{\mathbb{R}^d} (|\xi|^2 - 1)_-^\gamma\, d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + \frac{d}{2} + 1)}\,, \quad \gamma \geq 0\,.$$

Usually such formulae are obtained for a class of smooth potentials with compact supports. One of the major applications of the inequalities (1.2) is that they allow one to close the class of smooth compactly supported potentials in $L^{\gamma+d/2}(\mathbb{R}^d)$ and obtain Weyl's asymptotics (1.3) for arbitrary potentials providing that the right hand side in (1.3) is finite, see [**BirS**].

Clearly the asymptotic formula (1.3) immediately implies $L^{cl}_{\gamma,d} \leq L_{\gamma,d}$. However, the finiteness of the constants $L^{cl}_{\gamma,d}$ does not imply the finiteness of the constants $L_{\gamma,d}$. For example, it has been showen in [**BirL**] (see also [**LN**]) that if $d = 2$ and $\gamma = 0$, then the condition $V \in L^1(\mathbb{R}^2)$ does not guarantee (1.3). Moreover, it might happen that for the number of negative eigenvalues $N(\alpha V)$ of the operator H the asymptotics is of "Weyl's order" but the coefficient is not classical, namely,

$$\lim_{\alpha\to+\infty} \alpha^{-1} N(\alpha V) = \frac{1}{4\pi} \int_{\mathbb{R}^2} V_-(x)\, dx + \beta$$

with some $\beta > 0$.

1.3. Sharp values of the constants $L_{\gamma,1}$, $\gamma \geq 3/2$. The sharp values of $L_{\gamma,d}$ are known for $\gamma \geq 3/2$, $d = 1$, (see [**LT, AizL**]), where they coincide with $L^{cl}_{\gamma,d}$. For $\gamma = 3/2$ and $d = 1$ this fact follows from one of the so-called Buslaev-Faddeev-Zakharov trace formulae [**FaZ**] (see also [**BF**]) .

Assume that V is a smooth function such that $\operatorname{supp} V \in (-c, c)$ for some $c > 0$ and let us consider the equation

$$Hu(x,k) = -\frac{d^2}{dx^2} u(x,k) + V(x) u(x,k) = k^2 u(x,k)$$

where $\operatorname{Im} k \geq 0$ and u satisfies the conditions

$$u(x,k) = \begin{cases} e^{ikx}, & x > c \\ a(k) e^{ikx} + b(k) e^{-ikx}, & x < -c. \end{cases}$$

The Wronskian $W[u, \bar u]$ is constant if $k \in \mathbb{R}$ and comparing its values for $x > c$ and $x < -c$ we immediately find the magic formula

$$|a(k)|^2 - |b(k)|^2 = 1,$$

which, in particular, implies that $|a(k)| \geq 1$ for all $k \in \mathbb{R}$.

The negative eigenvalues $\{\lambda_n = \lambda_n(V)\}$ of the operator H can be parameterised by $(i\kappa_n)^2$, where $\kappa_n > 0$. Assuming that $u(x, i\kappa_n) \in L^2(\mathbb{R})$ is an eigenfunction of the operator H, we immediately obtain that this is possible if and only if the scattering coefficient $a(i\kappa_n) = 0$ and thus the eigenvalues of H are identified with zeros of $a(k)$ in upper complex plane.

In [**FaZ**] the authors have obtained infinite number of trace formulae relating the discrete spectrum of H, the scattering coefficient a and the values of integrals of the potential function V and its derivatives. The first three such formulae are given by

$$-\sum_n \kappa_n + \frac{1}{2\pi}\int_{-\infty}^{\infty} \log|a(k)|\,dk = \frac{1}{4}\int_{-\infty}^{\infty} V(x)\,dx,$$

$$\sum_n \kappa_n^3 + \frac{3}{2\pi}\int_{-\infty}^{\infty} k^2\,\log|a(k)|\,dk = \frac{3}{16}\int_{-\infty}^{\infty} V^2(x)\,dx, \tag{1.5}$$

$$-\sum_n \kappa_n^5 + \frac{5}{2\pi}\int_{-\infty}^{\infty} k^4\,\log|a(k)|\,dk = \frac{5}{32}\int_{-\infty}^{\infty} V^3(x)\,dx + \frac{5}{64}\int_{-\infty}^{\infty} (V'(x))^2\,dx.$$

Let now consider (1.5). By using the variational principle and the fact that $|a(k)| \geq 1$ on the real line we obtain

$$\sum_n |\lambda_n(V)|^{3/2} \leq \sum_n |\lambda_n(-V_-)|^{3/2} = \sum_n (\kappa_n(-V_-))^3 \leq \frac{3}{16}\int_{-\infty}^{\infty} (V_-(x))^2\,dx.$$

It remains to notice that the constant $L^{cl}_{3/2,1}$ appearing in (1.3) coincides with 3/16.

If $\gamma > 3/2$ we use an idea suggested by Aizenman and Lieb [**AizL**].
Denote by $B(p,q)$ the classical Beta function

$$\mathcal{B}(p,q) = \int_0^1 (1-t)^{q-1} t^{p-1}\,dt.$$

Then

$$\begin{aligned}
\sum_n |\lambda_n(V)|^\gamma &= \frac{1}{\mathcal{B}(\gamma-3/2,5/2)} \sum_n \int_0^\infty (|\lambda_n(V)|-t)_+^{3/2}\,t^{\gamma-3/2-1}\,dt \\
&= \frac{1}{\mathcal{B}(\gamma-3/2,5/2)} \sum_n \int_0^\infty (|\lambda_n(V+t)|)^{3/2}\,t^{\gamma-3/2-1}\,dt \\
&\leq \frac{1}{\mathcal{B}(\gamma-3/2,5/2)}\, L^{cl}_{3/2,1} \int_{-\infty}^{\infty}\int_0^\infty ((V(x)+t)_-)^2 t^{\gamma-3/2-1}\,dtdx \\
&= \frac{\mathcal{B}(\gamma-3/2,3)}{\mathcal{B}(\gamma-3/2,5/2)}\, L^{cl}_{3/2,1} \int_{-\infty}^{\infty} (V_-(x))^{\gamma+1/2}\,dx \\
&= L^{cl}_{\gamma,1} \int_{-\infty}^{\infty} (V_-(x))^{\gamma+1/2}\,dx.
\end{aligned}$$

1.4. Sharp values of the constants $L_{1/2,1}$. Hundertmark, Lieb and Thomas have shown in [**HLT**] that the sharp value of $L_{1/2,1}$ is equal to 1/2 which is twice the classical constant $L^{cl}_{1/2,1} = 1/4$. Then by applying the argument described above we find that $L_{\gamma,1} = 2L^{cl}_{\gamma,1}$, for $1/2 \leq \gamma < 3/2$.

Conjecture. *In their paper* [**LT**] *Lieb and Thirring have conjectured that* $L_{1,d} = L^{cl}_{1,d}$ *for* $d \geq 3$. *This conjecture is still open.*

Note that B.Helffer and D.Robert [**HR1, HR2**] have proved that the constant $L_{\gamma,d} > L^{cl}_{\gamma,d}$ for $\gamma < 1$.

1.5. Sharp values of the constants $L_{\gamma,d}$, $\gamma \ge 3/2$ and $d > 1$. In [**LW1**] A.Laptev and T.Weidl were able to obtain that

$$\sum_n |\lambda_n| \le L^{cl}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+d/2}\, dx$$

for any $\gamma \ge 3/2$ and $d \ge 1$, see the next Section.

Remark 1. *The sharp values of the constants $L_{\gamma,1}$ for $1/2 < \gamma < 3/2$; $L_{\gamma,2}$, $0 < \gamma < 3/2$ and $L_{\gamma,d}$, $0 \le \gamma < 3/2$, $d \ge 0$ are unknown.*

1.6. Bounds for special classed of potentials. For a class of potentials equal characteristic functions of sets of finite measure improved constants in CLR inequalities were obtained in [**Lap3**]. For example, it is proved that for this class of potential $L_{0,3} \le \frac{\sqrt{3}}{2\pi^2} 0.0877$. This is better than Lieb's bound which is about 0.1156. It is interesting that the constant 0.0877 is only slightly exceeds the constant appearing from the imbedded Sobolev theorem which for $d = 3$ equals $0.0780\dots$.

2. Matrix-valued potentials

2.1. Sharp inequalities in higher dimensions. In [**LW1**] A.Laptev and T.Weidl extended trace formulae for scalar Schrödinger operators to Schrödinger operators with Hermitian martix-valued potentials. Let

$$HU(x,k) = -\frac{d^2}{dx^2} U(x,k) + V(x) U(x,k) = k^2 U(x,k),$$

where V is a smooth $m \times m$ Hermitian martix-valued function such that $\operatorname{supp} V \subset (-c, c)$, $c > 0$. We choose U such that

$$U(x,k) = \begin{cases} I\, e^{ikx}, & x > c \\ A(k) e^{ikx} + B(k) e^{-ikx}, & x < -c. \end{cases}$$

Here I is the identity $m \times m$-matrix and A and B are "scattering" $m \times m$-matrices which are defined for $\operatorname{Im} k \ge 0$. If U^* is the matrix adjoint to U, then the Wronskian $W[U, U^*]$ is independent of the variable x and we obtain that

$$AA^* - BB^* = I,$$

which, in particular, implies $\det |A| \ge 1$. Moreover, in [**LW1**] the authors established that

$$\sum_n m_n \kappa_n^3 + \frac{3}{2\pi} \int_{-\infty}^{\infty} k^2 \log\det |A(k)|\, dk = \frac{3}{16} \int_{-\infty}^{\infty} \operatorname{Tr} V^2(x)\, dx,$$

where κ_n are eigenvalues of the operator H and m_n are their multiplicities. Thus if we count the eigenvalues $\{\lambda_n\}$ of the operator H together with their multiplicities then

$$\sum_n |\lambda_n|^{3/2} = \sum_n m_n \kappa_n^3 \le L^{cl}_{3/2,1} \int_{-\infty}^{\infty} \operatorname{Tr} V_-^2(x)\, dx.$$

(Note that this inequality was also obtained later in the paper of R.Benguria and M.Loss [**BL**].)

Applying the Aizenman-Lieb argument we find that for any $\gamma \ge 3/2$

$$\sum_n |\lambda_n|^\gamma \le L^{cl}_{\gamma,1} \int_{-\infty}^{\infty} \operatorname{Tr} V_-^{\gamma+1/2}(x)\, dx.$$

Finally by using the so-called "lifting argument with respect to dimensions" we are able to obtain sharp Lieb-Thirring inequalities for $\gamma \geq 3/2$ and $d \geq 1$. Indeed, let $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ and let us denote by Δ' the Laplacian with respect to $x' = (x_2, \dots, x_d)$. Then

$$\begin{aligned}
(2.1) \quad \sum_n |\lambda_n(-\Delta + V)|^\gamma &= \sum_n |\lambda_n(-\partial_{x_1}^2 - \Delta' + V)|^\gamma \\
&\leq \sum_n |\lambda_n(-\partial_{x_1}^2 + (-\Delta' + V)_-)|^\gamma \\
&\leq L_{\gamma,1}^{cl} \int_{-\infty}^{\infty} \mathrm{Tr}\,(-\Delta' + V)_-^{\gamma+1/2}\, dx_1.
\end{aligned}$$

Using this trick d times we arrive at

$$\sum_n |\lambda_n(-\Delta + V)|^\gamma \leq \left(\Pi_{l=0}^{d-1} L_{\gamma+l/2,1}^{cl}\right) \int_{\mathbb{R}^d} V_-^{\gamma+d/2}\, dx = L_{\gamma,d}^{cl} \int_{\mathbb{R}^d} V_-^{\gamma+d/2}\, dx.$$

2.2. Improved inequalities in higher dimensions for $1/2 \leq \gamma < 3/2$. In [**HLW**] the authors were able to extend the sharp result of the D.Hundertmark, E.H.Lieb and L.E.Thomas [**HLT**] for $\gamma = 1/2$ to matrix-valued potentials. Then using "lifting argument with respect to dimensions" one immediately obtains

$$(2.2) \qquad L_{\gamma,d} \leq 2L_{\gamma,d}^{\mathrm{cl}} \quad \text{for all} \quad 1 \leq \gamma < 3/2\,, \quad d \in \mathbb{N}\,,$$

$$(2.3) \qquad L_{\gamma,d} \leq 2L_{\gamma,d}^{\mathrm{cl}} \quad \text{for all} \quad 1/2 \leq \gamma < 3/2\,, \quad d = 1\,,$$

$$(2.4) \qquad L_{\gamma,d} \leq 4L_{\gamma,d}^{\mathrm{cl}} \quad \text{for all} \quad 1/2 \leq \gamma < 1\,, \quad d \geq 2\,.$$

For the important case $\gamma = 1$, $d = 3$ we have $L_{1,3} \leq 2L_{1,3}^{cl} < 0.013509$ compared with $L_{1,3} < 5.96677L_{1,3}^{cl} < 0.040303$ obtained in [**L2**] and its improvement $L_{1,3} < 5.21803L_{1,3}^{cl} < 0.035246$ obtained in [**BlSt**].
Note also that inequalities (2.2) - (2.4) on the constant $L_{\gamma,d}$ imply that $L_{1,d} \leq 2L_{1,d}^{cl} < L_{0,d}^{cl}$ as was conjectured in [**Rue**].
The same arguments as in [**LW1**] yield the inequalities (2.2) - (2.4) for Schrödinger operators with magnetic fields.

2.3. Further improved bounds for $1 \leq \gamma < 3/2$. Recently the estimates for the constant $L_{\gamma,d}$, for $1 \leq \gamma < 3/2$ were improved in the paper of Dolbeault, Laptev and Loss [**DLL**], where the authors found that

$$(2.5) \qquad L_{\gamma,d} \leq R\, L_{\gamma,d}^{cl},$$

where

$$R := \frac{2}{3\sqrt{3}} \times \left(\frac{2}{3\pi}\right)^{-1} = 1.8138\dots.$$

At the moment the estimate (2.5) is the best known for the values of γ from the interval $1 \leq \gamma < 3/2$. The authors used the approach of A. Eden and C. Foias [**EF**] who have obtained Lieb-Thirring inequalities via generalised Sobolev inequalities for a system of orthonormal functions. For $\gamma = 1$ such an approach is equivalent to Lieb-Thirring bounds for Schrödinger operators, see for the proof [**LT**].
In [**DLL**] the 1D approach of Eden and Foias was developed for orthonormal system of vector-functions and thus allowed us to obtain improved constants

for Schrödinger operators with matrix-valued potentials. With already developed technique from [**LW1**] this implies improved constants for all $d \geq 1$ and $1 \leq \gamma < 3/2$.

2.4. Improved bounds for $\gamma = 0$. The best constant $L_{0,3}$, which is close to the optimal one, was obtained in [**L1**] by using the Feynman-Kac formula and Jensens inequality (see also [**RS**]). Using Cwikel's method D.Hundertmark [**H**] has proved the CLR inequality for matrix-valued potentials and thus by using the "lifting" arguments from [**LW1**] improve Lieb's bound for large d's.
Later the CLR inequality for matrix-valued potentials were obtained in [**FLS**] with better constants by using the original method of E.H.Lieb [**L1**].

3. Some other recent results related to Lieb-Thirring bounds

3.1. Monotonicity of eigenvalue moments by J.Stubbe. Let, as in Subsection 1.2, $\lambda_n(\alpha)$ be the negative eigenvalues of the operator $H(\alpha) = -\Delta + \alpha V$. It has been proved by J. Stubbe in [**St**] that if $V \leq 0$ then the mapping

$$\alpha \to \alpha^{-(2+d/2)} \sum_n |\lambda_n|^2 \tag{3.1}$$

is non decreasing for all $\alpha > 0$ and thus for all α

$$\alpha^{-(2+d/2)} \sum_n |\lambda_n(\alpha)|^2 \leq L^{cl}_{2,d} \int_{\mathbb{R}^d} |V(x)|^{2+d/2}\, dx.$$

Letting $\alpha = 1$ we obtain sharp Lieb-Therring inequalities for $\gamma = 2$ and therefore for any $\gamma \geq 2$ and for any $d \geq 1$.

Remark 2. *The monotonicity property of the mapping* (3.1) *is an interesting fact which unfortunately is not true for $\gamma < 2$, see* [**St**].

3.2. Harmonic oscillator. Let $H = -\Delta + \sum_{k=1}^d \omega_k^2 x_k^2$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. It has been shown in [**DeB**] and [**Lap3**] that for the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of the the operator H we have

$$\sum_n (\lambda - \lambda_k)_+ \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \Big(\lambda - \sum_{k=1}^d \omega_k^2 x_k^2\Big)_+ \, dx.$$

This inequality could be interpreted as a Lieb-Thirring inequality for the negative eigenvalues for the Schrödinger operator with the potential

$$V = \sum_{k=1}^d \omega_k^2 x_k^2 - \lambda$$

an it gives a class of examples of Schrödinger operators for which the Lieb-Thirring conjecture is justified.

3.3. Lieb-Thirring inequalities for Magnetic operators. The lifting argument with respect to dimensioned described in Section 2 allows one to state that all the results concerning Lieb-Thirring inequalities given in Section 2 for the operator $-\Delta + V$ hold true even for the magnetic operators $(i\nabla + \mathcal{A})^2 + V$, where the vector-potential $\mathcal{A} \in L^2_{\rm loc}(\mathbb{R}^2)$. This is a corollary of the lifting argument with respect to dimension. Indeed, at every step we can gauge away the vector potential when arguing as in (2.1).

Let now $\{\lambda_n(B)\}$ be eigenvalues of a Schrödinger operator with constant magnetic field in $\mathbb{R}^2$

$$H_B = \Big(i\,\frac{\partial}{\partial x_1} + \frac{Bx_2}{2}\Big)^2 + \Big(i\,\frac{\partial}{\partial x_2} - \frac{Bx_1}{2}\Big)^2 + V(x), \qquad x = (x_1, x_2).$$

Recently R. Frank and R. Olofsson [**FO**] have proved that if $B > 0$ then

$$\sum_n |\lambda_n(B)| \le 3\,\frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))_-\, dx. \tag{3.2}$$

It is easy to see that

$$\frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))_-\, dx \le \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} (|\xi|^2 + V(x))_-\, d\xi dx.$$

Therefore, although the constant 3 appearing in the right hand side of (3.2) is not as good as the constant $R = 1.8138...$ which appears in (2.5), this bound contains a much more relevant information related to the magnetic structure of the operator H_B.

3.4. CLR inequalities for Schrödinger operators with Aharonov-Bohm magnetic field. Let us consider in $L^2(\mathbb{R}^2)$ the operator H_β

$$H_\beta = (i\nabla + \mathcal{A}_\beta)^2 + V(x), \tag{3.3}$$

where $V(x) = V(|x|)$ and

$$\mathcal{A}_\beta = \beta\Big(\frac{x_2}{|x|^2}, -\frac{x_1}{|x|}\Big), \qquad \beta \in (0,1).$$

It has been shown by A.A. Balinsky, W.D. Evans and R.T. Lewis in [**BEL**] that there is a constant $C = C(\beta) > 0$ such that for the number of the negative eigenvalues $N(H_\beta)$ the operator (3.3) the following CLR inequality holds

$$N(H_\beta) \le C \int_{\mathbb{R}^2} V_-(x)\, dx. \tag{3.4}$$

Here we shall give a prove of this bound with the optimal constant. Indeed, since V depends only on $|x|$, then by using polar coordinates we obtain that the quadratic form of the operator H_β equals

$$\int_{\mathbb{S}} \int_0^\infty \Big(|\partial_r u|^2 + \frac{(k-\beta)^2}{r^2}\,|u|^2 + V(r)|u|^2\Big)\, r\, dr d\theta, \quad k = 0, \pm 1, \pm 2, \dots.$$

Substituting $r = e^t$, $v(t,\theta) = u(e^t, \theta)$ we arrive to the quadratic form

$$\int_{\mathbb{S}} \int_{-\infty}^\infty \Big(|\partial_t v|^2 + (n-\beta)^2 |v|^2 + \tilde V(t)|v|^2\Big)\, dt d\theta,$$

where $\tilde V(t) = V(e^t)\,e^{2t}$. Let $\{-\nu_n\}$ be the negative eigenvalues of the 1D Schrödinger operator whose quadratic form equals

$$\int_{-\infty}^{\infty} (|v'(t)|^2 + \tilde V(t)|v(t)|^2)\,dt.$$

Then for the number of the negative eigenvalues below zero of the operator H_β we have

$$N(H_\beta) = \#\{k, n:\, -\nu_n + (k-\beta)^2 < 0,\, k\in\mathbb{Z},\, n\in\mathbb{N}\} \le R(\beta)\sum_n \nu_n^{1/2}, \tag{3.5}$$

where

$$R(\beta) = \sup_k \Big\{\nu^{-1/2}\cdot \big(\#\{k:\, -\nu + (k-\beta)^2 < 0,\, k\in\mathbb{Z}\}\big)\Big\}.$$

Using the sharp inequality of Hundertmark-Lieb-Thomas [**HLT**] we obtain

$$\sum_n \nu_n^{1/2} \le \frac12 \int_{-\infty}^{\infty} \tilde V(t)\,dt = \frac{1}{4\pi}\int_{\mathbb{R}^2} V(x)\,dx.$$

By using the latter inequality together with (3.5) we finally obtain

$$N(H_\beta) \le \frac{R(\beta)}{4\pi}\int_{\mathbb{R}^2} V(x)\,dx.$$

Note that since $k = 0\in\mathbb{Z}$ we have $R(\beta)\to\infty$ if $\beta\to 0$ or $\beta\to 1$.

REMARK 3. *Concluding this Section we would like to mention a recent result of M.Rumin* [**Rum**] *who was able to find a new interesting way of proving Lieb-Thirring inequalities.*

4. Applications of Lieb-Thirring inequalities

Lieb-Thirring inequalities have a large number of applications. Here we present some of them.

1. In their celebrated paper [**LT**] E.H.Lieb and W.Thirring obtained inequalities (1.2) in order to apply them for problems of stability of matter, see the recent excellent book [**LS**]. The most important case for physics is the case $\gamma = 1$, $d = 3$, where the sharp value of the constant $L_{1,3}$ is still unknown but any improvement of its bound is of great importance for Quantum Mechanics.

2. Typically the properties of the continuous spectrum of Schrödinger operators depend on the properties of the discrete spectrum. The negative part of potentials drags from the positive spectral semi-axis the discrete spectrum destroying its absolute continuity. In the most transparent way this has been shown for 1D Schrödinger operators in the paper of P.Deift and R.Killip [**DK**], where the authors proved that perturbation by potentials $V\in L^2(\mathbb{R})$ essentially preserve the absolute continuity of the spectral interval $(0,\infty)$. The corresponding conjecture of B.Simon [**S1**] which he suggested for multi-dimensional Schrödinger operators still remains open.

3. Lieb-Thirring inequalities proved to be very useful when estimating dimensions of attractors in theory of Navier-Stokes equations, see [**L4**].

4. Recently one more applications has been made. The best bound on the value of the constant $L_{1,3}$ has been used in the paper of P.T. Nam [**N**] for finding new bounds on the maximum ionization of atoms. Such bounds improve the result of E.H.Lieb [**L3**] in the fermionic case.

5. One more application was given in the paper of E.H. Lieb [**L5**], where the author considered bounds for the Riesz and Bessel potentials of orthonormal functions.

5. Dirichlet and Neumann boundary value problems

5.1. Weyl's asymptotics and Pólya's conjecture. Let $\Omega \in \mathbb{R}^d$, $d \geq 1$, be an open domain of finite Lebesgue measure, $|\Omega| < \infty$. Denote by $0 < \lambda_1 < \lambda_2 \leq \dots$ and $0 = \mu_1 < \mu_2 \leq \dots$ respectively the eigenvalues of the Dirichlet and Neumann boundary value problem for the Laplace operator in $L^2(\Omega)$

$$-\Delta^{\mathcal{D}} u = \lambda u, \qquad u|_{\partial\Omega} = 0. \tag{5.1}$$

$$-\Delta^{\mathcal{N}} u = \mu\, u, \qquad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0. \tag{5.2}$$

Let T be an operator with discrete spectrum $\{\lambda_n = \lambda_n(T)\}_{n=1}^{\infty}$, $\lambda_n \to \infty$, as $n \to \infty$, and let $N(\lambda, T)$ be its counting function of the spectrum

$$N(\lambda, T) = \#\{n : \lambda_n(T) < \lambda\}.$$

The study of the counting functions of the Dirichlet and Neumann eigenvalues has a rich history. It was conjectures by H.Weyl [**Weyl**] that

$$N(\lambda, -\Delta^{\mathcal{D}}) = L^{cl}_{0,d}\, |\Omega|\, \lambda^{d/2} - \frac{1}{4}\, L^{cl}_{0,d-1}\, |\partial\Omega|\, \lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}), \tag{5.3}$$

as $\lambda \to \infty$, and respectively

$$N(\mu, -\Delta^{\mathcal{N}}) = L^{cl}_{0,d}\, |\Omega|\, \mu^{d/2} + \frac{1}{4}\, L^{cl}_{0,d-1}\, |\partial\Omega|\, \mu^{(d-1)/2} + o(\mu^{(d-1)/2}), \tag{5.4}$$

as $\mu \to \infty$. Here we denote by $|\partial\Omega|$ the $(d-1)$-Lebesgue measure of the boundary of the domain Ω.

Important contributions in proving these formulae were made by Courant and Hilbert [**CH**], and later by L.Hörmander [**Ho**], H.Dustermaat and V.Guillemin [**DG**], R.Melrose [**Melr**], Yu.Safarov and D.Vassiliev [**SV**] and many others. Finally Weyl's conjecture has been proved by V.Ivrii in [**Ivr**] under some assumption on the measure of points generating closed billiards in Ω and smoothness of the boundary $\partial\Omega$.

Note that (5.3) and (5.4) imply the asymptotic formula for the so-called Riesz means of the eigenvalues

$$\sum_n (\lambda - \lambda_n)_+^{\gamma} \;=\; L^{cl}_{\gamma,d}\, |\Omega|\, \lambda^{\gamma+d/2} - \frac{1}{4}\, L^{cl}_{\gamma,d-1}\, |\partial\Omega|\, \lambda^{\gamma+(d-1)/2} + o(\lambda^{\gamma+(d-1)/2}), \quad \lambda \to \infty, \tag{5.5}$$

$$\sum_n (\mu - \mu_n)_+^{\gamma} \;=\; L^{cl}_{\gamma,d}\, |\Omega|\, \mu^{\gamma+d/2} + \frac{1}{4}\, L^{cl}_{\gamma,d-1}\, |\partial\Omega|\, \mu^{\gamma+(d-1)/2} + o(\mu^{\gamma+(d-1)/2}), \quad \mu \to \infty, \tag{5.6}$$

where $\gamma \geq 0$ and where if $\gamma = 0$ the left hand sides of (5.5) and (5.6) coincide with the values of $N(\lambda, -\Delta^{\mathcal{D}})$ and $N(\mu, -\Delta^{\mathcal{N}})$ respectively.

In 1961 G. Pólya [**P**] made a natural conjecture that there is the following uniform estimate for the counting function of the Dirichlet Laplacian in domains Ω of finite Lebesgue measure

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq \lambda^{d/2} L^{cl}_{0,d} |\Omega|, \quad \lambda > 0. \tag{5.7}$$

He proved it for the class of tiling domains, namely such domains whose infinite number of copies can fill the whole space $\mathbb{R}^d$ without any gaps by using translations and rotation.
Remarkably this conjecture still remains open even for such a simple domain as the disc $\{x \in \mathbb{R}^2 : |x| < 1\}$, where the eigenvalues of the Dirichlet Laplacians could be calculated via the roots of Bessel functions.

The only progress in proving Pólya's conjecture has been made in [**Lap1**], where it has been noticed that if $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, where $d_1 + d_2 = d$, $d_1 \geq 2$, $d_2 \geq 1$ and the operator of the Dirichlet boundary problem in $L^2(\Omega_1)$ satisfies the Pólya conjecture and Ω_2 is an arbitrary domain whose d_2-Lebesgue measure is finite, then

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq \lambda^{d/2} L^{cl}_{0,d,} |\Omega|, \qquad \lambda > 0,$$

or equivalently,

$$\lambda_k \geq (L^{cl}_{0,d} |\Omega|)^{-2/d} k^{2/d}, \qquad k \in \mathbb{N}.$$

5.2. Easier problems, Berezin-Li-Yau inequalities.

There are two somewhat easier problems related to the Pólya conjecture.
The first one concerns the prove of (5.7) for arbitrary domains of finite measure with some constant which could be larger than $L^{cl}_{0,d}$. Solving such a problem allows one to obtain the main term in the Weyl asymptotics for arbitrary domains of finite measure without any assumption on the smoothness of the boundary.

For bounded domains such a result was proved independently by Z.Ciesielski [**C**] and M.Sh.Birman and M.Z.Solomyak [**BirS**]. For arbitrary domains of finite Lebesgue measure this result is due to G.Rozenblum [**Roz2**] and E.H.Lieb [**L1**], see also [**Met**].

Another easier problem concerns inequalities for regularised traces, namely, inequalities for the Riesz means of the eigenvalues

$$\sum_n (\lambda - \lambda_n)_+^\gamma \leq L_{\gamma,d} \, \lambda^{\gamma + d/2} |\Omega|, \quad \lambda > 0, \tag{5.8}$$

where $\gamma \geq 0$. It is easy to show that if in (5.8) for some $\gamma_0 \geq 0$ we have $L_{\gamma_0,d} = L^{cl}_{\gamma_0,d}$, then $L_{\gamma,d} = L^{cl}_{\gamma,d}$ for any $\gamma > \gamma_0$.

P. Li and S.-T. Yau have proved in [**LY**] that for any $k \in \mathbb{N}$

$$\sum_{n=1}^k \lambda_n \geq \frac{d}{d+2} (L^{cl}_{0,d} |\Omega|)^{-2/d} k^{\frac{d+2}{d}} = (L^{cl}_{1,d} |\Omega|)^{-2/d} k^{\frac{d+2}{d}}, \tag{5.9}$$

where the constant $(L^{cl}_{1,d})^{-2/d}$ in the right hand side of (5.9) is sharp.

Using a similar approach P.Kröger [**K**] has obtained the sharp upper bound on the Neumann eigenvalues

$$\sum_{n=1}^{k} \mu_n \leq (L^{cl}_{1,d}\,|\Omega|)^{-2/d}\, k^{\frac{d+2}{d}}, \qquad \forall k \in \mathbb{N}. \tag{5.10}$$

It has been pointed out in [**LW2**] that by using the Legendre transform such inequalities are equivalent to the inequalities essentially obtained by F.Berezin in [**Ber**]

$$\sum_n (\lambda - \lambda_n)_+ \leq L^{cl}_{1,d}\, \lambda^{1+d/2}|\Omega|, \quad \lambda > 0 \tag{5.11}$$

and

$$\sum_n (\mu - \mu_n)_+ \geq L^{cl}_{1,d}\, \mu^{1+d/2}|\Omega|, \quad \mu > 0. \tag{5.12}$$

By using (5.11) and (5.12) one can always "go down" and obtain bounds for the Riesz means with $0 \leq \gamma < 1$, see [**Lap1**]. However, usually one is not able to keep sharp constants. In particular, (5.11) and (5.12) imply a bound for the number of the eigenvalues with the best currently known constants

$$N(\lambda, -\Delta^{\mathcal{D}}) \leq \lambda^{d/2} L^{cl}_{0,d} \Big(1 + \frac{2}{d}\Big)^{d/2} |\Omega| \tag{5.13}$$

and

$$N(\mu, -\Delta^{\mathcal{N}}) \geq \mu^{d/2} L^{cl}_{0,d} \frac{2}{d+2}\, |\Omega|. \tag{5.14}$$

'Going up" in γ is a lot better. Indeed, for example the bound (5.11) implies bounds for higher moment without loosing the sharpness of the constants. Namely,

$$\sum_n (\lambda - \lambda_n)^\gamma_+ \leq L^{cl}_{\gamma,d}\, \lambda^{\gamma+d/2}|\Omega|, \quad \lambda > 0, \quad \gamma \geq 1. \tag{5.15}$$

REMARK 4. *The bounds* (5.15) *could be interpreted as Lieb-Thirring inequalities for the number of negative eigenvalues of Schrödinger operators with a class of barrier type potentials*

$$V_\lambda(x) = \begin{cases} -\lambda, & x \in \Omega, \\ +\infty & x \notin \Omega. \end{cases}$$

5.3. Two terms inequalities for the Riesz means of the eigenvalues of Dirichlet Laplacians.

Recently, several improvements of the inequality (5.15) have been found. Initially they have been obtained for the discrete Laplace operator, see [**FLU**]. The first result for the continuous Laplacian is due to Melas [**Mel**]. From his work follows that

$$\sum_n (\lambda - \lambda_n)^\gamma_+ \leq\, L^{cl}_{\gamma,d}\,|\Omega| \Big(\lambda - M_d \frac{|\Omega|}{I(\Omega)}\Big)^{\gamma+d/2}_+, \quad \lambda > 0\,, \quad \gamma \geq 1\,, \tag{5.16}$$

where M_d is a constant depending only on the dimension and $I(\Omega)$ denotes the second moment of Ω, see also [**Ilyin, Yol**] for further generalisations. One should mention, however, that these corrections do not capture the correct order in λ from the second term of Weyl's asymptotics.

In the case $\gamma \geq 3/2$ it is known, [**W2**], that one can strengthen (5.15) for any open set $\Omega \subset \mathbb{R}^d$ with a negative remainder term reflecting the correct order in λ in comparison to the second term of (5.3). However, since one can increase $|\partial\Omega|$ without changing the individual eigenvalues λ_k significantly, a direct analog of the first two terms of the asymptotics (5.3) cannot yield a uniform bound on the eigenvalue means. Therefore any uniform improvement of (5.15) without any further conditions on Ω must invoke other geometric quantities.

The main result from [**W2**] states that the remainder term involves certain projections on $d-1$-dimensional hyperplanes. In [**GW**] a universal improvement of (5.15) has been found. It holds for $\gamma \geq 3/2$ with a correction term of correct order depending only on the volume of Ω.

5.4. Most recent bounds with two terms.
In the recent paper [**GLW**] the authors obtain a number of results for the values $\gamma \geq 3/2$. Here is one of them: let $\Omega \subset \mathbb{R}^d$ be a bounded, convex domain with smooth boundary and assume that the curvature of $\partial\Omega$ is bounded from above by $1/R$. Then for all $\lambda > 0$

$$\sum_n (\lambda - \lambda_n)_+^\gamma \leq L_{\gamma,d}^{cl}\, |\Omega| \lambda^{\gamma+d/2} - L_{\gamma,d}^{cl}\, 2^{-d-2}\, |\partial\Omega|\, \lambda^{\gamma+(d-1)/2} \int_0^1 \Big(1 - \frac{d-1}{4R\sqrt{\lambda}}\, s\Big)_+ \, ds.$$

The proof of this bound involves a one dimensional sharp inequality for the eigenvalues of the operator $-d^2/dx^2$ with Dirichlet boundary conditions on the interval $(0,l)$. Obviously the eigenvalues of this operator are equal to $j^2\pi^2/l^2, j = 1,2,3\ldots$. The authors show that

$$\sum_j \Big(\lambda - \frac{j^2\pi^2}{l^2}\Big)_+ \leq L_{1,1}^{cl} \int_0^l \Big(\lambda - \frac{1}{4\delta^2(t)}\Big)\, dt,$$

where $\delta(t) = \min\{t, l-t\}$. It is interesting that this inequality is sharp and it involves the term $1/4\delta^2$ which usually appears in Hardy's inequalities.

5.5. Spectral inequalities for Dirichlet Laplacians with constant magnetic field. Let $H_{0,B}$ the Dirichlet Laplacian with constant magnetic field acting in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a domain of finite Lebesgue measure,

$$H_{0,B} = \Big(i\,\frac{\partial}{\partial x_1} + \frac{Bx_2}{2}\Big)^2 + \Big(i\,\frac{\partial}{\partial x_2} - \frac{Bx_1}{2}\Big)^2. \tag{5.17}$$

Let $\{\lambda_n(B)\}_{n=1}^\infty$ be the eigenvalues of $H_{0,B}$. It has been proved in [**ELV**] that these eigenvalues satisfy the Berezin-Li-Yau inequality with $\gamma \geq 1$

$$\sum_n (\lambda - \lambda_n(B))_+^\gamma \leq \frac{1}{(2\pi)^2}\, |\Omega| \int_{\mathbb{R}^2} (\lambda - |\xi|)_+^\gamma \, d\xi, \qquad \lambda > 0.$$

In the paper by R.Frank, M.Loss and T.Weidl [**FLW**] this result has been extended to $0 \leq \gamma < 1$

$$\sum_n (\lambda - \lambda_n(B))_+^\gamma \leq R_\gamma\, L_{\gamma,2}^{cl}\, \lambda^{\gamma+1}. \tag{5.18}$$

where

$$R_\gamma = 2\left(\frac{\gamma}{\gamma+1}\right)^\gamma, \qquad 0 < \gamma < 1,$$

and $R_0 = 2$. The authors proved that the constants $R_\gamma > 1$, $0 \le \gamma < 1$, are sharp and cannot be made smaller even for tiling domains. Moreover, the authors were able to find further improvement of the inequality (5.18) taking into account the values of the Landau levels

$$\sum_n (\lambda - \lambda_n(B))_+^\gamma \le \frac{B}{2\pi}\,|\Omega| \sum_{k=0}^\infty (\lambda - B(2k+1))^\gamma,$$

where it is assumed that $0^0 = 0$.

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