SPECTRAL ASYMPTOTIC BEHAVIOR OF A CLASS OF INTEGRAL OPERATORS

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Integral operators of the type

$$(\mathbf{T}f)(x)=\int_0^1 \frac{x^{\beta}y^{\gamma}}{(x+y)^{\alpha}} f(y)\,dy,$$

the kernels of which have a singularity at a single point, are discussed. H. Widom's method and some of his results are used to show that, if $\alpha>0,\ \beta,\ \gamma>-\frac{1}{2},\ \rho\overset{\text{def}}{=}\beta+\gamma-\alpha+1>0$, then we have for the distribution function of the singular numbers of the operator,

$$\lim_{\varepsilon\to 0} N(\varepsilon, \mathbf{T}) \ln^{-2}\frac{1}{\varepsilon}=\frac{1}{2\pi^2\rho}.$$

1. As a rule, when the asymptotic behavior of the eigenvalues of an integral operator has been discussed, kernels which have a singularity on a diagonal have been considered (see e.g., [1], where a bibliography may be found). The asymptotic behavior is here of a power type and is only influenced by the behavior of the kernel in the neighborhood of the diagonal.

Conditions of a different type were imposed on the kernel by H. Widom [2]. In [2] was investigated the asymptotic behavior of the eigenvalues of integral operators of the convolution type, under the assumption that the Fourier transform $\widetilde{K}(t)$ of the kernel is a positive even function, decreasing for $t>0$. The asymptotic behavior is expressed in terms of the behavior of $\widetilde{K}(t)$ as $t\to\infty$ and is not necessarily of the power type.

In the present article we examine the asymptotic behavior of the singular numbers* (s-numbers) of the integral operator

$$(\mathbf{T}f)(x)=\int_0^1 \frac{x^{\beta}y^{\gamma}}{(x+y)^{\alpha}} f(y)\,dy, \tag{1}$$

acting in space $L_2(0, 1)$. We shall obtain the principal term of the asymptotic form of the s-numbers of the operator (1). Since the kernel has a singularity only at the point (0, 0), the asymptotic behavior proves to be of the hyperpower type. It is influenced solely by the neighborhood of the point (0, 0), so that the result would be unaffected were the "weight" x^{β}, y^{γ} in (1) to be replaced by arbitrary functions $u(x)$, $v(y)$ such that $u(x)\sim c_1x^{\beta}$, $v(y)\sim c_2y^{\gamma}$ as $x, y\to 0$. Let us denote by $N(\varepsilon, \mathbf{K}; (a, b)\times(c, d))$ the distribution function of the s-numbers of an integral operator $\mathbf{K}$, acting from $L_2(a, b)$ into $L_2(c, d)$:

$$N(\varepsilon, \mathbf{K})=\sum\nolimits_{n:s_n(\mathbf{K})>\varepsilon} 1.$$

*Regarding the singular numbers of completely continuous operators see [3].

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Our basic result is as follows:

THEOREM. Let

$$\alpha > 0;\ \beta, \gamma > -\frac{1}{2};\quad \rho \overset{\text{def}}{=} \beta + \gamma - \alpha + 1 > 0. \tag{2}$$

Then the following asymptotic formula holds for the s-numbers of the integral operator (1):

$$\lim_{\varepsilon \to 0} (\ln 1/\varepsilon)^{-2} N(\varepsilon, T) = (2\pi^2\rho)^{-1}. \tag{3}$$

With $\beta = \gamma$ the operator (1) is symmetric and positive, so that (3) transforms into the relation for the eigenvalues. If at the same time $\alpha = 1$, we can obtain (3) from Widom's theorem [see [4], Theorem (3.3)].

2. Widom's method [4] may be used for proving our theorem. Considerable technical modifications have to be made as compared with [4], because the kernels considered are more complicated. Below we give a statement of Widom's lemma [2], on which the proof of our theorem is based. This lemma concerns the behavior of the eigenvalues of the integral operator

$$(G_{\nu,a}f)(x) = \int_0^a \frac{\sin(\nu(x-y))}{\pi(x-y)} f(y)\,dy, \tag{4}$$

acting in space $L_2(0, a)$. The quadratic form of the operator (4) can be simply expressed in terms of the Fourier transform

$$F(t) = \int_0^a f(x) e^{itx} dx. \tag{5}$$

We have, in fact,

$$(G_{\nu,a}f, f) = \frac{1}{2\pi}\int_{-\nu}^{\nu} |F(t)|^2\,dt.$$

LEMMA 1. The following statements can be made regarding the eigenvalues of the integral operator (4):

1) given an arbitrary $\delta > 0$, there exists a number C_δ such that with $\nu a \le (1-\delta)\pi n$ we have

$$\lambda_n(G_{\nu,a}) < e^{-C_\delta n};$$

2) with arbitrary δ, $\varepsilon > 0$, there exists $N = N_{\delta,\varepsilon}$ such that with $n > N$, $\nu a \ge (1+\delta)\pi n$ we have

$$1 - \varepsilon < \lambda_n(G_{\nu,a}) < 1.$$

We shall use Lemma 1 for two-sided estimates of the eigenvalues of integral operators of a special type.

Let the function $K(x)$, $-\infty < x < \infty$, be such that its Fourier transform

$$\tilde{K}(t) = \int_{-\infty}^{\infty} K(x) e^{-itx} dx$$

is an even nonnegative function, monotonically decreasing for $t > 0$ and such that

$$\tilde{K}(t) = e^{-At + o(t)}, \qquad t \to \infty. \tag{6}$$

Denote by $\tilde{K}^{-1}(\zeta)$, $\zeta > 0$, the function inverse to $\tilde{K}(t)$, $t > 0$.

Consider the integral operator

$$(K_a f)(x) = \int_0^a K(x-y) f(y)\,dy, \tag{7}$$

acting in $L_2(0, a)$.

LEMMA 2. Let the kernel of the operator (7) satisfy the above conditions. Then, given any $\delta > 0$, there exist C'_δ, C''_δ, $a_\delta > 0$ such that with arbitrary $a \geqslant a_\delta$, $\varepsilon > 0$ we have

$$\frac{a}{A\pi(1+\delta)} \ln \frac{C'_\delta}{\varepsilon} \leqslant N(\varepsilon, \mathbf{K}_a) \leqslant \frac{a}{A\pi(1-\delta)} \ln \frac{C''_\delta}{\varepsilon}. \tag{8}$$

Proof. We transform the quadratic form of the operator $\mathbf{K}_a$:

$$(\mathbf{K}_a f, f) = \int_0^a \int_0^a K(x-y) \overline{f(x)} f(y)\, dy\, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{K}(t) |F(t)|^2 dt, \tag{9}$$

where the function F is defined by (5). Since $\tilde{K}$ is monotonic, with arbitrary $\nu > 0$ we have

$$(\mathbf{K}_a f, f) \leqslant \frac{\tilde{K}(0)}{2\pi} \int_{-\nu}^{+\nu} |F(t)|^2 dt + \frac{\tilde{K}(\nu)}{2\pi} \int_{-\infty}^{+\infty} |F(t)|^2 dt = \tilde{K}(0)(\mathbf{G}_{\nu,a} f, f) + \tilde{K}(\nu)(f, f).$$

We fix $\delta > 0$. Let $\nu a = (1 - \delta/2)\pi n$; then, by Lemma 1, Paragraph 1, and the familiar properties of the s-numbers,

$$\lambda_n(\mathbf{K}_a) \leqslant \tilde{K}(0) e^{-nC_{\delta/2}} + \tilde{K}((1-\delta/2)\pi a^{-1} n). \tag{10}$$

In accordance with (6), there exists C'_δ, $C'_\delta \geq 2K(0)$, such that

$$\tilde{K}((1-\delta/2)\pi a^{-1} n) \leqslant \frac{C'_\delta}{2} e^{-(1-\delta)A\pi a^{-1} n}.$$

We put $a_\delta = \pi A C_{\delta/2}^{-1}$. Then with $a \geqslant a_\delta e^{-nC_{\delta/2}} < e^{-(1-\delta)A\pi a^{-1} n}$ we finally obtain

$$\lambda_n(\mathbf{K}_a) \leqslant C'_\delta e^{-(1-\delta)A\pi a^{-1} n}.$$

This last is equivalent to the right-hand inequality of (8). To prove the left-hand inequality, we use the relation

$$(\mathbf{K}_a f, f) > \tilde{K}(\nu)(\mathbf{G}_{\nu,a} f, f),$$

whence

$$\lambda_n(\mathbf{K}_a) > \tilde{K}(\nu) \lambda_n(\mathbf{G}_{\nu,a}).$$

Let $\nu a = (1+\delta/2)\pi n$. Then, by Lemma 1, Paragraph 2, with $\varepsilon > 0$ there exists N such that when $n > N$, $\lambda_n(G_{\nu,a}) > 1 - \varepsilon$. If $\theta = \min_{n \leqslant N} (\lambda_n(\mathbf{G}_{\nu,a}), \sqrt{1-\varepsilon})$, we find by virtue of the properties of the function $\tilde{K}(\nu)$ that there exists C''_δ, $\theta^2 > C''_\delta > 0$, such that for any positive integer n

$$\lambda_n(\mathbf{G}_{\nu,a}) > \sqrt{C''_\delta}, \quad \tilde{K}(\nu) > \sqrt{C''_\delta} e^{-A(1+\delta)\pi a^{-1} n},$$

which proves the left-hand inequality of (8). Q.E.D.

3. Let us transform the operator (1) to a form more convenient for our investigation.

To this end, consider the operator

$$Vf = g, \quad g(x) = f(e^{-2x}) e^{-x} \sqrt{2}, \tag{11}$$

isometrically mapping $L_2(0, 1)$ into $L_2(0, \infty)$. The operator $\hat{\mathbf{T}} = VTV^*$, unitarily equivalent to the operator $\mathbf{T}$, is given by the relation

$$(\hat{\mathbf{T}} g)(x) = 2^{1-\alpha} \int_0^{\infty} e^{(\alpha-1-2\beta)x} e^{(\alpha-1-2\gamma)y} \operatorname{ch}^{-\alpha}(x-y)\, g(y)\, dy. \tag{12}$$

The kernel of this operator [as distinct from the kernel of the operator (1)] contains a function, dependent on the difference between the arguments. The idea of the further proof is as follows: we divide the semi-axis $[0, \infty)$ into intervals, in each of which we replace the "weight" of the type e^{cx} by a constant; in each of the "diagonal" squares we employ inequalities derived from Lemma 3, and we estimate the error caused by the presence of the "nondiagonal" squares. This leads to (3).

We still require some auxiliary propositions. The first is obtained by applying Lemma 2 to a kernel of special type. Denote by $\mathbf{K}_{\alpha,a}$ the integral operator in $L_2(0, a)$ given by

$$(\mathbf{K}_{\alpha,a}f)(x) = 2^{1-\alpha}\int_0^a \operatorname{ch}^{-\alpha}(x-y)f(y)\,dy, \quad 0<\alpha. \tag{13}$$

The Fourier transform of the kernel is found from the expression (see [5])

$$\widetilde{K}_\alpha(t) = \frac{1}{2\Gamma(\alpha+1)}\left|\Gamma\left(\frac{\alpha}{2}+\frac{it}{2}\right)\right|^2. \tag{14}$$

Hence $\widetilde{K}_\alpha$ is an even positive function. It can easily be shown (e.g., by expanding the function into an infinite product) that $\widetilde{K}_\alpha$ is monotonically decreasing for $t \geq 0$.

We next use Binet's equation (see [6]) for the principal branch of $\log\Gamma(z)$:

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \int_{-\infty}^{\infty}\left(\frac{1}{2}-\frac{1}{x}+\frac{1}{e^x-1}\right)\frac{e^{-zx}}{x}\,dx, \quad \operatorname{Re} z>0.$$

From this and (13), we have

$$\log\widetilde{K}(t) = -\frac{\pi}{2}t + o(t). \tag{15}$$

The kernel of the integral operator (13) thus satisfies the conditions of Lemma 2 (with $A = \pi/2$), and hence inequalities of the type (8) hold for the distribution function of the eigenvalues of the operator (13).

For estimating the errors caused by the presence of the "nondiagonal" squares, we have to consider operators in $L_2(0, \infty)$ of the type

$$(\mathbf{K}f)(x) = 2^{1-\alpha}\int_0^\infty \operatorname{ch}^{-\alpha}(x+y)\,e^{-(\theta-\alpha)(x+y)}f(y)\,dy. \tag{16}$$

<u>LEMMA 3.</u> Let $\theta > 0$. Then, for the s-numbers of the integral operator (16) we have the bound

$$N(\varepsilon, \mathbf{K}) \leqslant C_1 \ln\frac{1}{\varepsilon}, \text{ where } \varepsilon<1. \tag{17}$$

<u>Proof.</u> Denote by l the positive integer such that $l\theta \geq 1$, and introduce the auxiliary operator

$$Uf = F, \quad F(x) = \sqrt{\frac{l}{2}}\,f\left(\frac{l}{2}\ln 1/x\right)x^{-1/2},$$

isometrically mapping $L_2(0, \infty)$ into $L_2(0, 1)$. The s-numbers of the operator $\mathbf{K}$ are equal to the s-numbers of the operator $\hat{\mathbf{K}} = \mathbf{U}\mathbf{K}\mathbf{U}^*$, where

$$(\hat{\mathbf{K}}F)(x) = l\int_0^1 (xy)^{(l\theta-1)/2}(1+(xy)^l)^{-\alpha}F(y)\,dy.$$

The integral operator with the kernel $(1+x^l y^l)^{-\alpha}$ satisfies the conditions of A. O. Gel'fond's theorem (see [7], Theorem II), so that its eigenvalues will have order $O(e^{-c_1 n})$. The operator of multiplication by the function $x^{(l\theta-1)/2}$ in space $L_2(0, 1)$ has norm unity. Hence we obtain (17).

<u>4. Proof of the Theorem.</u> We fix $a > 0$ and a positive integer m. We define a set of operators $\mathbf{P}_j$, Q_j in space $L_2(0, \infty)$. Let $\mathbf{P}_j$, $\mathbf{Q}_j$ be the operators of multiplication by the characteristic functions of the sets $[(j-1)a, ja)$, $[ja, \infty)$ $(j = 1, \ldots, m)$, respectively.

We use these operators to decompose the operator (12) into the sum

$$\hat{\mathbf{T}} = \sum_{j=1}^m \mathbf{P}_j\hat{\mathbf{T}}\mathbf{P}_j + \sum_{j=1}^m(\mathbf{Q}_j\hat{\mathbf{T}}\mathbf{P}_j + \mathbf{P}_j\hat{\mathbf{T}}\mathbf{Q}_j) + \mathbf{Q}_m\mathbf{T}\mathbf{Q}_m. \tag{18}$$

Returning by means of the change of variables $x \mapsto -\frac{1}{2}e^{-ma}\ln x$, $y \mapsto -\frac{1}{2}e^{-ma}\ln y$ to the interval of integration [0, 1], we find that the operator $Q_m\hat{T}Q_m$ is unitarily equivalent to the operator $e^{-2\rho ma}T$, where T is the initial operator (1) and ρ is defined by (2).

Consequently,

$$\|Q_m T Q_m\| \leqslant C_2 e^{-2\rho ma}, \quad C_2 = \|T\|.$$

Given $\varepsilon > 0$ and a, let the number m be chosen from the condition

$$2C_2 e^{-2\rho ma} < \varepsilon \leqslant 2C_2 e^{-2\rho(m-1)a}. \tag{19}$$

Then,

$$N(\varepsilon/2, Q_m\hat{T}Q_m) = 0.$$

Further, using the familiar properties of the s-numbers, we find that

$$N\left(\varepsilon, \sum_{j=1}^{m} P_j\hat{T}P_j\right) = \sum_{j=1}^{m} N(\varepsilon, P_j\hat{T}P_j) \leqslant \sum_{j=1}^{m} N(\varepsilon, e^{-2\rho(j-1)a+qa}K_{\alpha,a}),$$

where $q = |\alpha - 1 - 2\beta| + |\alpha - 1 - 2\gamma|$.

By (15) and the right-hand inequality of (8), given any $\delta > 0$, there exist a_δ, C_δ'' such that with $a > a_\delta$, $\varepsilon > 0$ we have

$$N\left(\varepsilon, \sum_{j=1}^{m} P_j\hat{T}P_j\right) \leqslant \frac{2a}{\pi^2(1-\delta)} \sum_{j=1}^{m} \ln \frac{e^{qa}C_\delta''}{\varepsilon \cdot e^{2\rho(j-1)a}}. \tag{20}$$

From the inequalities (19) we obtain a bound for the product ma, where one of these parameters, say a, can be specified arbitrarily. If we set

$$a = a(\varepsilon) = \ln^{1/2} \frac{2C_2}{\varepsilon}, \tag{21}$$

then

$$m = \frac{1}{2\rho} \ln^{1/2} \frac{2C_2}{\varepsilon} + o\left(\ln^{1/2} \frac{1}{\varepsilon}\right). \tag{22}$$

Then, given a_δ, there exists $\varepsilon_0 > 0$ such that with $\varepsilon < \varepsilon_0$ we have $a(\varepsilon) \geq a_\delta$, and hence (20) remains in force.

Using (21) and (22), we easily obtain the limit relation

$$\lim_{\varepsilon \to 0} \ln^{-2} \frac{1}{\varepsilon} \cdot a(\varepsilon) \cdot \sum_{j=1}^{m} \ln \frac{e^{qa(\varepsilon)}C_\delta''}{\varepsilon \cdot e^{-2\rho(j-1)a(\varepsilon)}} = \frac{1}{4\rho}.$$

Hence,

$$\overline{\lim_{\varepsilon \to 0}} \ln^{-2} \frac{1}{\varepsilon} \cdot N\left(\varepsilon, \sum_{j=1}^{m} P_j\hat{T}P_j\right) \leqslant \frac{1}{2\pi^2\rho(1-\delta)}.$$

A similar lower bound may be found from (21) and (22) and the left-hand inequality of (8); we obtain

$$\underline{\lim_{\varepsilon \to 0}} \ln^{-2} \frac{1}{\varepsilon} N\left(\varepsilon, \sum_{j=1}^{m} P_j\hat{T}P_j\right) \geqslant \frac{1}{2\pi^2\rho(1+\delta)}.$$

Consider the distribution functions of the operators $Q_j\hat{T}P_j$, where $1 \leq j \leq m$.

On multiplying the kernel of the operator (12) by $e^{2\rho y}$ and using simple working, we find that

$$N(\varepsilon, Q_j\hat{T}P_j) \leqslant N(\varepsilon, 2^{1-\alpha}e^{-(2\beta+1-\alpha)(x-y)}\operatorname{ch}^{-\alpha}(x-y), ((j-1)a, ja)(ja, \infty)) \leqslant N(\varepsilon, 2^{1-\alpha}e^{-(2\beta+1-\alpha)(x+y)}\operatorname{ch}^{-\alpha}(x+y), (x, y \geqslant 0)).$$

Since $2\beta > -1$, we have by Lemma 3

$$N(\varepsilon, Q_j\hat{T}P_j) \leqslant C_1 \ln \frac{1}{\varepsilon}.$$

The same inequality holds for the operators $P_j\hat{T}Q_j$ ($1 \le j \le m$). Using the Ky Fan inequality

$$N(\varepsilon_1 + \varepsilon_2,\ A + B) \leqslant N(\varepsilon_1,\ A) + N(\varepsilon_2, B),$$

we obtain

$$N(\varepsilon, \hat{T}) \leqslant N\left(\frac{\varepsilon}{4}, \sum_{j=1}^{m} P_j\hat{T}P_j\right) + \sum_{j=1}^{m}\left(N\left(\frac{\varepsilon}{8m}, Q_j\hat{T}P_j\right) + N\left(\frac{\varepsilon}{8m}, P_j\hat{T}Q_j\right)\right). \tag{23}$$

Multiplying (23) by $\ln^{-2}(1/\varepsilon)$ and passing to the limit superior, we find that

$$\overline{\lim_{\varepsilon \to 0}} \ln^{-2}\frac{1}{\varepsilon} \cdot N(\varepsilon, \hat{T}) \leqslant \frac{1}{(1-\delta)\, 2\pi^2\rho}. \tag{24}$$

Similarly,

$$N(\varepsilon, \hat{T}) \geqslant N\left(\frac{7}{4}\varepsilon, \sum_{j=1}^{m} P_j\hat{T}P_j\right) - \sum_{j=1}^{m}\left(N\left(\frac{\varepsilon}{8m}, Q_j\hat{T}P_j\right) + N\left(\frac{\varepsilon}{8m}, P_j\hat{T}Q_j\right)\right),$$

and, on passing to the limit inferior,

$$\underline{\lim_{\varepsilon \to 0}} \ln^{-2}\frac{1}{\varepsilon} \cdot N(\varepsilon, \hat{T}) \geqslant \frac{1}{(1+\delta)\, 2\pi^2\rho}. \tag{25}$$

The inequalities (24) and (25), which hold for arbitrary $\delta > 0$, prove the theorem.

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