WEYL ASYMPTOTICS FOR PERTURBED FUNCTIONAL DIFFERENCE OPERATORS

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Abstract. We consider the difference operator $H_W = U + U^{-1} + W$, where U is the self-adjoint Weyl operator $U = \mathrm{e}^{-bP}$, $b > 0$, and the potential W is of the form $W(x) = x^{2N} + r(x)$ with $N \in \mathbb{N}$ and $|r(x)| \leq C(1 + |x|^{2N-\varepsilon})$ for some $0 < \varepsilon \leq 2N - 1$. This class of potentials W includes polynomials of even degree with leading coefficient 1, which have recently been considered in [3]. In this paper we show that such operators have discrete spectrum and obtain Weyl-type asymptotics for the Riesz means and for the number of eigenvalues. This is an extension of the result previously obtained in [8] for $W = V + \zeta V^{-1}$, where $V = \mathrm{e}^{2\pi bx}$, $\zeta > 0$.

1. Introduction

The mirror manifolds of toric Calabi–Yau manifolds can be described by algebraic curves and recently [2] it was observed that quantisation of these curves leads to functional-difference operators. For local del Pezzo Calabi–Yau threefolds the simplest example yields the operator $H_{CY} = U + U^{-1} + V + \zeta V^{-1}$ on $L^2(\mathbb{R})$ [5]. Here, U and V denote the self-adjoint Weyl operators $U = \mathrm{e}^{-bP}$ and $V = \mathrm{e}^{2\pi bQ}$ for $b > 0$, where $(P\psi)(x) = \mathrm{i}\psi'(x)$ and $(Q\psi)(x) = x\psi(x)$ are the quantum mechanical momentum and position operators on $L^2(\mathbb{R})$.

In a recent paper [8], the authors proved that for $\zeta > 0$, the operator H_{CY} has a self-adjoint extension with purely discrete spectrum consisting of finite multiplicity eigenvalues tending to infinity. In addition, Weyl-type asymptotics for the Riesz means and for the number of eigenvalues were established. These results prove that H_{CY}^{-1} is trace-class, which confirms part of a conjecture in [2]. In this short note, we consider the difference operator

$$H_W = U + U^{-1} + W = H_0 + W$$

where $W \in \mathcal{C}(\mathbb{R})$ is a continuous, real-valued potential satisfying $\lim_{|x|\to\infty} W(x) = \infty$ that is of the form $W(x) = x^{2N} + r(x)$ with $N \in \mathbb{N}$ and $|r(x)| \leq C(1 + |x|^{2N-\varepsilon})$ for some $0 < \varepsilon \leq 2N - 1$. This class of potentials includes polynomials of even degree with leading coefficient 1, which have recently been considered in [3]. We will prove that such an operator admits a self-adjoint extension with discrete eigenvalues λ_j of finite multiplicity converging to infinity. Subsequently we will prove Weyl-type asymptotics for the Riesz mean $\sum_{j\geq 1}(\lambda - \lambda_j)_+$ and for the number of eigenvalues below a given value λ as $\lambda \to \infty$. (Note that the Riesz means for the negative spectrum of Schrödinger operators with decaying potentials are associated with Lieb–Thirring inequalities [10].) These results prove that, if H_W is invertible, the inverse H_W^{-1} is trace-class, as claimed in [3]. Our proof method also applies to the previously considered potential $V + \zeta V^{-1}$.

2. Main Results

Since $W \in \mathcal{C}(\mathbb{R})$ with $\lim_{|x|\to\infty} W(x) = \infty$, we can conclude that W is bounded from below. As a consequence, the symmetric operator $H = H_0 + W$ is bounded from below on the common domain of H_0 and W. We can thus consider its self-adjoint Friedrichs extension, which we continue to denote by H_W. The following proposition was proved in [8].

Proposition 1. [8, Proposition 2.1] *Let $W(x)$ be a continuous, real-valued, bounded below function such that $\lim_{|x|\to\infty} W(x) = +\infty$. Then the operator $H_W = H_0 + W$ has purely discrete spectrum consisting of finite multiplicity eigenvalues tending to $+\infty$.*

In [8] we considered the potential $W(x) = 2\cosh(2\pi bx)$ and proved the following result.

Theorem 2. [8, Theorem 2.2] *For the eigenvalues λ_j of $H_0 + 2\cosh(2\pi bx)$ it holds that*

$$\lim_{\lambda\to\infty} \frac{\sum_{j\geq 1}(\lambda-\lambda_j)_+}{\iint_{\mathbb{R}^2}(\lambda - 2\cosh(2\pi bk) - 2\cosh(2\pi bx))_+ \,\mathrm{d}k\,\mathrm{d}x} = \lim_{\lambda\to\infty} \frac{\sum_{j\geq 1}(\lambda-\lambda_j)_+}{\frac{1}{(\pi b)^2}\lambda\log^2\lambda} = 1$$

with a lower order term of the form $O(\lambda\log\lambda)$.

Here $a_+ = (a+|a|)/2$ denotes the positive part of a real variable. We also obtained the asymptotics for the number of eigenvalues smaller than λ.

Corollary 3. [8, Corollary 2.3] *For the number of eigenvalues $N(\lambda) = \#\{j \geq 1 : \lambda_j < \lambda\}$ of $H_0 + 2\cosh(2\pi bx)$ smaller than λ it holds that*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\iint_{\lambda - 2\cosh(2\pi k) - W(x)\geq 0} \mathrm{d}k\,\mathrm{d}x} = \lim_{\lambda\to\infty} \frac{N(\lambda)}{\frac{1}{(\pi b)^2}\log^2\lambda} = 1\,.$$

In this short note, we will prove the following analogous result for potentials growing polynomially.

Theorem 4. *Let $N \in \mathbb{N}$ and let $r \in \mathcal{C}(\mathbb{R})$ be a function such that $|r(x)| \leq C(1+|x|^{2N-\varepsilon})$ for some $C > 0$, $2N - 1 \geq \varepsilon > 0$ and all $x \in \mathbb{R}$. For the eigenvalues λ_j of $H_0 + x^{2N} + r(x)$ it holds that*

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{\sum_{j\geq 1}(\lambda-\lambda_j)_+}{\iint_{\mathbb{R}^2}(\lambda - 2\cosh(2\pi bk) - x^{2N} - r(x))_+ \,\mathrm{d}k\,\mathrm{d}x} \\
&= \lim_{\lambda\to\infty} \frac{\sum_{j\geq 1}(\lambda-\lambda_j)_+}{\iint_{\mathbb{R}^2}(\lambda - 2\cosh(2\pi bk) - x^{2N})_+ \,\mathrm{d}k\,\mathrm{d}x} = \lim_{\lambda\to\infty} \frac{\sum_{j\geq 1}(\lambda-\lambda_j)_+}{\frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda} = 1
\end{aligned}$$

with a lower order term of the form $O(\lambda^{\frac{2N+1}{2N}})$.

Corollary 5. *Let $N \in \mathbb{N}$ and let $r \in \mathcal{C}(\mathbb{R})$ be a function such that $|r(x)| \leq C(1+|x|^{2N-\varepsilon})$ for some $C > 0$, $2N - 1 \geq \varepsilon > 0$ and all $x \in \mathbb{R}$. For the number of eigenvalues $N(\lambda) = \#\{j \geq 1 : \lambda_j < \lambda\}$ of $H_0 + x^{2N} + r(x)$ smaller than λ it holds that*

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{N(\lambda)}{\iint_{\lambda - 2\cosh(2\pi bk) - x^{2N} - r(x)\geq 0} \mathrm{d}k\,\mathrm{d}x} \\
&= \lim_{\lambda\to\infty} \frac{N(\lambda)}{\iint_{\lambda - 2\cosh(2\pi bk) - x^{2N}\geq 0} \mathrm{d}k\,\mathrm{d}x} = \lim_{\lambda\to\infty} \frac{N(\lambda)}{\frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda} = 1.
\end{aligned}$$

Remark 6. Let $H_0 + x^{2N} + r_N(x)$ satisfy the assumptions of Theorem 4. If the operator is invertible, as is for example the case if $x^{2N} + r_N(x) \geq c > -2$, then we can repeat the argument of [8] to prove that $(H_0 + x^{2N} + r(x))^{-1}$ is trace-class. Assuming for simplicity that $\lambda_1 > 0$, this follows from

$$\sum_{j\geq 1}\frac{1}{|\lambda_j|} = \int_{\lambda_1}^{\infty}\frac{1}{\lambda}\,\mathrm{d}N(\lambda) = \left.\frac{N(\lambda)}{\lambda}\right|_{\lambda_1}^{\infty} + \int_{\lambda_1}^{\infty}\frac{N(\lambda)}{\lambda^2}\,\mathrm{d}\lambda < \infty\,.$$

Remark 7. The total symbol of the operator $H_0 + W$ is given by

$$\sigma(x,k) = 2\cosh(2\pi bk) + W(x)\,.$$

Theorem 2 and Theorem 4 as well as Corollary 3 and Corollary 5 are Weyl-type results that link the asymptotic behaviour of quantum mechanical expressions to classical phase space integrals.

In Section 3 we will give a proof of Theorem 2 and Theorem 4. It follows our arguments in [8], where the special case of $W(x) = 2\cosh(2\pi bx)$ was considered. In Section 4 we will prove Corollary 3 and Corollary 5. Before we proceed with the proofs, we compute the explicit asymptotics of the phase space integrals. This will prove equality between the limits in the results above.

2.1. **Leading order terms for hyperbolic cosine potential.** In [8] we computed that for any $C, D > 0$

$$\iint_{\mathbb{R}^2} (\lambda - 2D\cosh(2\pi bk) - 2C\cosh(2\pi bx))_+ \,\mathrm{d}k\,\mathrm{d}x = \frac{\lambda \log^2 \lambda}{(\pi b)^2} + O(\lambda \log \lambda) \tag{1}$$

as $\lambda \to \infty$ and similarly

$$\iint_{\lambda - 2D\cosh(2\pi bk) - 2C\cosh(2\pi bx) \geq 0} \mathrm{d}k\,\mathrm{d}x = \frac{\log^2 \lambda}{(\pi b)^2} + o(\log^2 \lambda) \tag{2}$$

as $\lambda \to \infty$.

2.2. **Leading order terms for polynomial potential.** Let $N \in \mathbb{N}$. By the min-max principle it is sufficient to consider $r(x) = C(1 + |x|^{2N-\varepsilon})$ with $C \in \mathbb{R}$ and prove that the asymptotic behaviour of the phase space integrals is independent of C as $\lambda \to \infty$. For later reference, we will also include an additional multiplication factor $D > 0$ in front of the term $2\cosh(2\pi bk)$ and prove that the asymptotic behaviour does not depend on D.

Using the symmetry of the integrand as well as $2\cosh(2\pi bk) \geq \mathrm{e}^{2\pi bk}$ for $k > 0$ together with the substitution $u = D\mathrm{e}^{2\pi bk}$ we can compute that

$$\begin{aligned}
&\iint_{\mathbb{R}^2} (\lambda - 2D\cosh(2\pi bk) - x^{2N} - C(1 + |x|^{2N-\varepsilon}))_+ \,\mathrm{d}k\,\mathrm{d}x \\
&\leq 4\int_0^\infty \int_0^\infty (\lambda - D\mathrm{e}^{2\pi bk} - x^{2N} - C(1 + x^{2N-\varepsilon}))_+ \,\mathrm{d}k\,\mathrm{d}x \\
&= \frac{2}{\pi b}\int_D^\infty \int_0^\infty \frac{(\lambda - u - x^{2N} - C(1 + x^{2N-\varepsilon}))_+}{u} \,\mathrm{d}x\,\mathrm{d}u\,.
\end{aligned}$$

Substituting $v_1\lambda = u$ and $v_2\lambda^{1/(2N)} = x$ we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^2} (\lambda - 2D\cosh(2\pi bk) - x^{2N} - C(1 + |x|^{2N-\varepsilon}))_+ \,\mathrm{d}k\,\mathrm{d}x \\
&\leq \frac{2\lambda^{\frac{2N+1}{2N}}}{\pi b}\int_{D/\lambda}^\infty \int_0^\infty \frac{(1 - v_1 - v_2^{2N} - C\lambda^{-1} - Cv_2^{2N-\varepsilon}\lambda^{-\frac{\varepsilon}{2N}})_+}{v_1} \,\mathrm{d}v_2\,\mathrm{d}v_1\,.
\end{aligned}$$

Note that the domain of integration can be restricted to $v_2 \geq 0$ and $1 - v_1 - v_2^{2N} - C\lambda^{-1} - Cv_2^{2N-\varepsilon}\lambda^{-\frac{\varepsilon}{2N}} \geq 0$. We now use that $(a-b)_+ \leq a_+ + |b|$ for any $a, b \in \mathbb{R}$ to split the integral into two parts

$$\begin{aligned}
&\iint_{\mathbb{R}^2} (\lambda - 2D\cosh(2\pi bk) - x^{2N} - C(1 + |x|^{2N-\varepsilon}))_+ \,\mathrm{d}k\,\mathrm{d}x \\
&\leq \frac{2\lambda^{\frac{2N+1}{2N}}}{\pi b}\Bigg(\int_{D/\lambda}^1 \int_0^\infty \frac{(1 - v_1 - v_2^{2N})_+}{v_1} \,\mathrm{d}v_2\,\mathrm{d}v_1 \\
&\quad + |C|\int_{D/\lambda}^\infty \int_{v_2 \geq 0, 1 - v_1 - v_2^{2N} - C\lambda^{-1} - C\lambda^{-\frac{\varepsilon}{2N}} v_2^{2N-\varepsilon} \geq 0} \frac{\lambda^{-1} + v_2^{2N-\varepsilon}\lambda^{-\frac{\varepsilon}{2N}}}{v_1} \,\mathrm{d}v_2\,\mathrm{d}v_1\Bigg).
\end{aligned}$$

The first integral yields the correct leading order term. To prove this, we first compute the inner integral explicitly

$$\frac{2\lambda^{\frac{2N+1}{2N}}}{\pi b}\int_{D/\lambda}^{1}\int_{0}^{\infty}\frac{(1-v_1-v_2^{2N})_+}{v_1}\,\mathrm{d}v_2\,\mathrm{d}v_1=\frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\int_{D/\lambda}^{1}\frac{(1-v_1)^{\frac{2N+1}{2N}}}{v_1}\,\mathrm{d}v_1\,.$$

Using partial integration we then observe that

$$\int_{D/\lambda}^{1}\frac{(1-v_1)^{\frac{2N+1}{2N}}}{v_1}\,\mathrm{d}v_1=\log\Big(\frac{\lambda}{D}\Big)\Big(1-\frac{D}{\lambda}\Big)^{\frac{2N+1}{2N}}+\frac{2N+1}{2N}\int_{D/\lambda}^{1}(1-v_1)^{\frac{1}{2N}}\log v_1\,\mathrm{d}v_1\,.$$

It remains to note that

$$\left|\int_{D/\lambda}^{1}(1-v_1)^{\frac{1}{2N}}\log v_1\,\mathrm{d}v_1\right|=\int_{D/\lambda}^{1}(1-v_1)^{\frac{1}{2N}}\log\Big(\frac{1}{v_1}\Big)\,\mathrm{d}v_1\le\Big(1-\frac{D}{\lambda}\Big)^{\frac{1}{2N}}\Big(1-\frac{D}{\lambda}-\frac{D}{\lambda}\log\Big(\frac{\lambda}{D}\Big)\Big)$$

to establish the asymptotic behaviour

$$\frac{2\lambda^{\frac{2N+1}{2N}}}{\pi b}\int_{D/\lambda}^{1}\int_{0}^{\infty}\frac{(1-v_1-v_2^{2N})_+}{v_1}\,\mathrm{d}v_2\,\mathrm{d}v_1=\frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda+O(\lambda^{\frac{2N+1}{2N}})$$

as $\lambda\to\infty$. For the second integral, we use that for sufficiently large λ

$$1-v_1-v_2^{2N}-C\lambda^{-1}-Cv_2^{2N-\varepsilon}\lambda^{-\frac{\varepsilon}{2N}}\le 2-v_1-\frac{1}{2}v_2^{2N}$$

and that

$$\begin{aligned}\int_{D/\lambda}^{\infty}\int_{v_2\ge 0,2-v_1-\frac{1}{2}v_2^{2N}\ge 0}\frac{\lambda^{-1}+v_2^{2N-\varepsilon}\lambda^{-\frac{\varepsilon}{2N}}}{v_1}\,\mathrm{d}v_2\,\mathrm{d}v_1&\le\lambda^{-\frac{\varepsilon}{2N}}\int_{D/\lambda}^{2}\int_{0}^{2^{1/N}}\frac{5}{v_1}\,\mathrm{d}v_2\,\mathrm{d}v_1\\&\le 10\lambda^{-\frac{\varepsilon}{2N}}(\log\lambda-\log D+\log 2)\end{aligned}$$

since $\lambda^{-1+\frac{\varepsilon}{2N}}+v_2^{2N-\varepsilon}\le 5$ on the domain of integration for sufficiently large λ. Putting everything together and using that $\lim_{\lambda\to\infty}(\lambda^{\frac{2N+1-\varepsilon}{2N}}\log\lambda)/\lambda^{\frac{2N+1}{2N}}=0$ since $\varepsilon>0$, we obtain that

$$\begin{aligned}&\iint_{\mathbb{R}^2}(\lambda-2D\cosh(2\pi bk)-x^{2N}-C(1+|x|^{2N-\varepsilon}))_+\,\mathrm{d}k\,\mathrm{d}x\\&\le\frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda+O(\lambda^{\frac{2N+1}{2N}})\end{aligned}$$

as $\lambda\to\infty$. Similarly, we can use the fact that $2\cosh(2\pi bk)\le 2\mathrm{e}^{2\pi bk}$ for $k>0$ together with the substitution $u=2D\mathrm{e}^{2\pi bk}$ to obtain the lower bound

$$\begin{aligned}&\iint_{\mathbb{R}^2}(\lambda-2D\cosh(2\pi bk)-x^{2N}-C(1+|x|^{2N-\varepsilon}))_+\,\mathrm{d}k\,\mathrm{d}x\\&\ge 4\int_0^{\infty}\int_0^{\infty}(\lambda-2D\mathrm{e}^{2\pi bk}-x^{2N}-C(1+x^{2N-\varepsilon}))_+\,\mathrm{d}k\,\mathrm{d}x\\&=\frac{2}{\pi b}\int_{2D}^{\infty}\int_0^{\infty}\frac{(\lambda-u-x^{2N}-C(1+x^{2N-\varepsilon}))_+}{u}\,\mathrm{d}x\,\mathrm{d}u\,,\end{aligned}$$

and by similar arguments as above

$$\begin{aligned}&\iint_{\mathbb{R}^2}(\lambda-2D\cosh(2\pi bk)-x^{2N}-C(1+|x|^{2N-\varepsilon}))_+\,\mathrm{d}k\,\mathrm{d}x\\&\ge\frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda+O(\lambda^{\frac{2N+1}{2N}})\end{aligned}$$

as $\lambda \to \infty$. Together with the upper bound, we obtain that

$$\begin{aligned} &\iint_{\mathbb{R}^2} (\lambda - 2\cosh(2\pi bk) - x^{2N} - C(1+|x|^{2N-\varepsilon}))_+ \,\mathrm{d}k\,\mathrm{d}x \\ &= \frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda + O(\lambda^{\frac{2N+1}{2N}})\,. \end{aligned} \tag{3}$$

Similarly, we can show that as $\lambda \to \infty$

$$\iint_{\lambda - 2D\cosh(2\pi bk) - x^{2N} - C(1+|x|^{2N-\varepsilon}) \geq 0} \mathrm{d}k\,\mathrm{d}x = \frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda + o(\lambda^{\frac{1}{2N}}\log\lambda)\,.$$

3. The Proof of Theorems 2 and 4

To establish upper and lower bounds on the sum of the eigenvalues we will employ ideas from [6], with the Fourier transform replaced by the coherent state transform, which we will introduce below.

Let g_β be the Gaussian function $g_\beta(x) = (\beta/\pi)^{1/4}\mathrm{e}^{-\frac{\beta}{2}x^2}$ with some $\beta > 0$, fixed for the moment. Clearly g_β satisfies $\|g_\beta\|_2 = 1$ in $L^2(\mathbb{R})$. For $\psi \in L^2(\mathbb{R})$ the classical coherent state transform is given by

$$\widetilde{\psi}(k,y) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi \mathrm{i}kx} g_\beta(x-y)\psi(x)\,\mathrm{d}x = \langle \psi, e_{k,y}\rangle \tag{4}$$

where $e_{k,y} = \mathrm{e}^{2\pi \mathrm{i}k} g_\beta(x-y)$. Following the computations in [8], we can show that

$$\iint_{\mathbb{R}^2} d_\beta 2\cosh(2\pi k)|\widetilde{\psi}(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y = \int_{\mathbb{R}} (H_0\psi)(x)\overline{\psi(x)}\,\mathrm{d}x$$

where $d_\beta = \mathrm{e}^{-\beta b^2/4} < 1$. A standard computation furthermore shows that

$$\int_{\mathbb{R}^2} W(y)|\widetilde{\psi}(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y = \int_{\mathbb{R}} (W * g_\beta^2)(x)|\psi(x)|^2\,\mathrm{d}x\,.$$

For convenience, we set $W_\beta := W * g_\beta^2$. Note that $W_\beta \in \mathcal{C}(\mathbb{R})$ with $\inf_{x\in\mathbb{R}} W_\beta(x) \geq \inf_{x\in\mathbb{R}} W(x)$ since g_β is non-negative and $\|g_\beta\|_2 = 1$. Furthermore, $\lim_{|x|\to\infty} W_\beta(x) = \infty$ and thus the statements above on the self-adjoint extension and discreteness of the spectrum also hold for the operator $H_0 + W_\beta$.

We now establish results for the special case of a hyperbolic cosine potential as well as monomial potentials. In [8, Section 2.1] we proved the following.

Proposition 8. *Let $b > 0$. Then*

$$(2\cosh(2\pi bx) * g_\beta^2)(x) = \frac{1}{c_\beta} 2\cosh(2\pi bx)$$

where $c_\beta = \mathrm{e}^{-(\pi b)^2/\beta}$. Conversely,

$$2\cosh(2\pi bx) = (c_\beta 2\cosh(2\pi bx) * g_\beta^2)(x)\,.$$

An analogous result holds for monomials.

Proposition 9. *Let $N \in \mathbb{N}$. Then*

$$(x^{2N} * g_\beta^2)(x) = x^{2N} + p_N(x)$$

where $p_N(x) = \sum_{j=0}^{N-1} \frac{1}{\beta^{N-j}} a_{N,j} x^{2j}$ is an even polynomial of order $2N-2$ with coefficients $a_{N,j} \in \mathbb{R}_+$ independent of β. Conversely,

$$x^{2N} = \big((x^{2N} + q_N) * g_\beta^2\big)(x)$$

where $q_N(x) = \sum_{j=0}^{N-1} \frac{1}{\beta^{N-j}} b_{N,j} x^{2j}$ is an even polynomial of order $2N-2$ with coefficients $b_{N,j}$ independent of β.

Furthermore, for $0 < \varepsilon \leq 2N-1$ *it holds that*

$$(|x|^{2N-\varepsilon} * g_\beta^2)(x) \leq c_{\varepsilon,N}(1+|x|^{2N-\varepsilon})$$

with some constant $c_{\varepsilon,N} \geq 0$ *depending on* β *and conversely*

$$|x|^{2N-\varepsilon} \leq (|x|^{2N-\varepsilon} * g_\beta^2)(x)\,.$$

Proof. Writing

$$\int_{\mathbb{R}} (x-y)^{2N} g_\beta(y)^2 \,\mathrm{d}y = \sum_{k=0}^{2N} \binom{2N}{k} x^k \int_{\mathbb{R}} y^{2N-k} g_\beta(y)^2 \,\mathrm{d}y$$

and noting that the integral vanishes for odd k, we compute that

$$\int_{\mathbb{R}} (x-y)^{2N} g_\beta(y)^2 \,\mathrm{d}y = \sum_{j=0}^{N} \binom{2N}{2j} x^{2j} \int_{\mathbb{R}} y^{2N-2j} g_\beta(y)^2 \,\mathrm{d}y = x^{2N} + \frac{(2N)!}{2^{2N}\beta^N} \sum_{j=0}^{N-1} \frac{4^j \beta^j}{(N-j)!(2j)!} x^{2j}$$

which yields the first result. The second statement then follows by induction from the observations that by the first identity

$$x^2 = (x^2 * g_\beta^2)(x) - \frac{1}{2\beta} = \big((x^2 - 1/(2\beta)) * g_\beta^2\big)(x)$$

as well as

$$x^{2N} = (x^{2N} * g_\beta^2)(x) - p_N(x) = (x^{2N} * g_\beta^2)(x) - \sum_{j=0}^{N-1} \frac{1}{\beta^{N-j}} a_{N,j} x^{2j}\,.$$

Since $g_\beta^2 \geq 0$ and

$$|x-y|^{2N-\varepsilon} \leq \max(2|x|, 2|y|)^{2N-\varepsilon} \leq 2^{2N-\varepsilon}(|x|^{2N-\varepsilon} + |y|^{2N-\varepsilon})$$

we obtain the bound

$$\int_{\mathbb{R}} |x-y|^{2N-\varepsilon} g_\beta(y)^2 \,\mathrm{d}x \leq 2^{2N-\varepsilon} \left(|x|^{2N-\varepsilon} + \int_{\mathbb{R}} |y|^{2N-\varepsilon} g_\beta(y)^2 \,\mathrm{d}y \right)$$

which yields the claimed inequality. Finally, since $x \mapsto |x|^{2N-\varepsilon}$ is a convex function for $0 < \varepsilon \leq 2N-1$ and $\|g_\beta\|_2 = 1$ we can apply Jensen's inequality to obtain

$$\int_{\mathbb{R}} |x-y|^{2N-\varepsilon} g_\beta(y)^2 \,\mathrm{d}y \geq \left| \int_{\mathbb{R}} (x-y) g_\beta(y)^2 \,\mathrm{d}y \right|^{2N-\varepsilon} = |x|^{2N-\varepsilon}\,.$$

□

3.1. **Lower bound on the Riesz mean.** In [6] a lower bound on the eigenvalues of a general class of operators on sets of finite measure with Neumann boundary condition was proved by means of an argument that relied on the Fourier transform. Here, we use a similar approach, with the coherent state transform replacing the Fourier transform.

Let ψ_j^W denote the orthonormal eigenfunctions corresponding to the eigenvalues λ_j^W of the operator $H_0 + W$ satisfying the assumptions in Proposition 1. They form a complete set of the Hilbert space $L^2(\mathbb{R})$ and by Plancherel's theorem it holds that

$$\iint_{\mathbb{R}^2} |\widetilde{\psi}_j^{W_\beta}(k,y)|^2 \,\mathrm{d}k \,\mathrm{d}y = \left\| \psi_j^{W_\beta} \right\|_2^2 = 1\,. \tag{5}$$

We can thus write

$$\sum_{j\geq 1} (\lambda - \lambda_j^W)_+ = \sum_{j\geq 1} (\lambda - \lambda_j^W)_+ \iint_{\mathbb{R}^2} |\widetilde{\psi}_j(k,y)|^2 \,\mathrm{d}k \,\mathrm{d}y,$$

and inserting the definition (4)

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j^W)_+ &= \iint_{\mathbb{R}^2}\sum_{j\geq 1}(\lambda-\lambda_j^W)_+\langle\psi_j^W, e_{k,y}\rangle\overline{\langle\psi_j^W, e_{k,y}\rangle}\,\mathrm{d}k\,\mathrm{d}y\\ &= \iint_{\mathbb{R}^2}\sum_{j\geq 1}(\lambda-\lambda_j^W)_+\big\langle\langle e_{k,y},\psi_j^W\rangle\psi_j^W, e_{k,y}\big\rangle\,\mathrm{d}k\,\mathrm{d}y\,.\end{aligned}$$

We can replace the sum by an integral with respect to the projection-valued measure $\mathrm{d}E_\mu^W$ for H_0+W on $\mathbb{R}$ as

$$\sum_{j\geq 1}(\lambda-\lambda_j^W)_+ = \iint_{\mathbb{R}^2}\int_{\mathbb{R}}(\lambda-\mu)_+\langle\mathrm{d}E_\mu^W e_{k,y}, e_{k,y}\rangle\,\mathrm{d}k\,\mathrm{d}y\,.$$

By the spectral theorem

$$\int_{\mathbb{R}}\langle\mathrm{d}E_\mu^W e_{k,y}, e_{k,y}\rangle = \langle e_{k,y}, e_{k,y}\rangle = \|g\|_2^2 = 1,$$

and thus we can apply Jensen's inequality with the convex function $x\mapsto(\lambda-x)_+$ to obtain the lower bound

$$\sum_{j\geq 1}(\lambda-\lambda_j^W)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda-\int_{\mathbb{R}}\mu\langle\mathrm{d}E_\mu^W e_{k,y}, e_{k,y}\rangle\right)_+\mathrm{d}k\,\mathrm{d}y\,. \tag{6}$$

Finally, by the spectral theorem the inner integral is

$$\int_{\mathbb{R}}\mu\langle\mathrm{d}E_\mu^W e_{k,y}, e_{k,y}\rangle = \langle H_0 e_{k,y}, e_{k,y}\rangle + \langle W e_{k,y}, e_{k,y}\rangle\,.$$

Following [8] we know

$$\langle H_0 e_{k,y}, e_{k,y}\rangle = \frac{1}{d_\beta}2\cosh(2\pi k)$$

and we compute

$$\langle W e_{k,y}, e_{k,y}\rangle = \int_{\mathbb{R}}W(x)g_\beta(x-y)^2\,\mathrm{d}x = W_\beta(y)\,.$$

Combining these two results with (6) we arrive at

$$\sum_{j\geq 1}(\lambda-\lambda_j^W)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda-\frac{1}{d_\beta}2\cosh(2\pi bk)-W_\beta(y)\right)_+\mathrm{d}k\,\mathrm{d}y\,.$$

3.2. **Lower bound for hyperbolic cosine potential.** Let λ_j be the eigenvalues of $H_0+2\cosh(2\pi bx)$. In this special case, which was considered in [8], one obtains from the computations above with $W(x)=2\cosh(2\pi bx)$ and from Proposition 8 that

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda-\frac{1}{d_\beta}2\cosh(2\pi bk)-\frac{1}{c_\beta}2\cosh(2\pi by)\right)_+\mathrm{d}k\,\mathrm{d}y\,.$$

By (2) the asymptotic behaviour of this lower bound does not depend on c_β, d_β and is of the desired form.

3.3. **Lower bound for polynomial potential.** Let λ_j be the eigenvalues of $H_0+x^{2N}+r(x)$. By the min-max principle, we obtain a lower bound on the Riesz mean if we replace $x^{2N}+r(x)$ by the larger potential $x^{2N}+C(1+|x|^{2N-\varepsilon})$. Using the computations above with $W(x)=x^{2N}+C(1+|x|^{2N-\varepsilon})$ together with Proposition 9 and the fact that for some $0<\delta\leq\varepsilon$ both $|p_N(y)|/(1+|y|^{2N-\delta})$ and $(1+|y|^{2N-\varepsilon})/(1+|y|^{2N-\delta})$ are bounded, yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+\geq\iint_{\mathbb{R}^2}\left(\lambda-\frac{1}{d_\beta}2\cosh(2\pi bk)-y^{2N}-C_N(1+|y|^{2N-\delta})\right)_+\mathrm{d}k\,\mathrm{d}y$$

for some constant C_N depending on β. By (3) the asymptotic behaviour of this lower bound does not depend on C_N, d_β and is of the desired form.

3.4. **Upper bound on the Riesz mean.** Again, we will use a similar approach to [6], where an upper bound on the eigenvalues of a general class of operators on sets of finite measure with Dirichlet boundary condition was proved.

Let W be a potential that satisfies the assumptions of Proposition 1. Consider the eigenvalues $\lambda_j^{W_\beta}$ of H_0+W_β where $W_\beta=W*g_\beta^2$ and let $\psi_j^{W_\beta}$ be the corresponding orthonormal eigenfunctions. Using the coherent state representations established at the beginning of this section, we can write

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j^{W_\beta})_+&=\sum_{j\geq 1}(\lambda-\langle(H_0+W_\beta)\psi_j^{W_\beta},\psi_j^{W_\beta}\rangle)_+\\&=\sum_{j\geq 1}\left(\lambda-\iint_{\mathbb{R}^2}\big(d_\beta 2\cosh(2\pi bk)+W(y)\big)|\widetilde{\psi}_j^{W_\beta}(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y\right)_+.\end{aligned}$$

By (5) we can apply Jensen's inequality with the convex function $x\mapsto(\lambda-x)_+$ to obtain

$$\sum_{j\geq 1}(\lambda-\lambda_j^{W_\beta})_+\leq\iint_{\mathbb{R}^2}\big(\lambda-d_\beta 2\cosh(2\pi bk)-W(y)\big)_+\sum_{j\geq 1}|\widetilde{\psi}_j^{W_\beta}(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y\,.$$

The eigenfunctions $\psi_j^{W_\beta}$ form an orthonormal basis in $L^2(\mathbb{R})$ and thus for all $k,y\in\mathbb{R}$

$$\sum_{j\geq 1}|\widetilde{\psi}_j^{W_\beta}(k,y)|^2=\sum_{j\geq 1}|\langle e_{k,y},\psi_j^{W_\beta}\rangle|^2=\|e_{k,y}\|^2=1$$

which yields the upper bound

$$\sum_{j\geq 1}(\lambda-\lambda_j^{W_\beta})_+\leq\iint_{\mathbb{R}^2}\big(\lambda-d_\beta 2\cosh(2\pi bk)-W(y)\big)_+\,\mathrm{d}k\,\mathrm{d}y\,.$$

3.5. **Upper bound for hyperbolic cosine potential.** Let λ_j be the eigenvalues of $H_0+2\cosh(2\pi bx)$. In this special case, which was considered in [8], we can choose $W(x)=c_\beta 2\cosh(2\pi bx)$ such that $W_\beta(x)=2\cosh(2\pi bx)$ by Proposition 8. The computation above then yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+\leq\iint_{\mathbb{R}^2}\big(\lambda-d_\beta 2\cosh(2\pi bk)-c_\beta 2\cosh(2\pi by)\big)_+\,\mathrm{d}k\,\mathrm{d}y\,.$$

By (2) the asymptotic behaviour of this upper bound does not depend on c_β, d_β and is of the desired form.

3.6. **Upper bound for polynomial potential.** Let λ_j be the eigenvalues of $H_0+x^{2N}+r(x)$. By the min-max principle, we obtain an upper bound on the Riesz mean if we replace $x^{2N}+r(x)$ by the smaller potential $x^{2N}-C(1+|x|^{2N-\varepsilon})$. Applying again the min-max principle together with the last statement in Proposition 9, we may further decrease this potential to $x^{2N}-C\big((1+|x|^{2N-\varepsilon})*g_\beta^2\big)(x)$. By Proposition 9 this potential coincides with W_β for the choice $W(x)=x^{2N}+q_N(x)-C(1+|x|^{2N-\varepsilon})$.

The computation above and the fact that for some $0 < \delta \leq \varepsilon$ both $|q_N(y)|/(1+|y|^{2N-\delta})$ and $(1+|y|^{2N-\varepsilon})/(1+|y|^{2N-\delta})$ are bounded, yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \iint_{\mathbb{R}^2}\left(\lambda - d_\beta 2\cosh(2\pi bk) - y^{2N} + C_N(1+|y|^{2N-\delta})\right)_+ \mathrm{d}k\,\mathrm{d}y$$

with a constant C_N depending on β. By (3) the asymptotic behaviour of this upper bound does not depend on C_N, d_β and is of the desired form.

4. The Proof of Corollaries 3 and 5

In [8] we provided two proofs of Corollary 3. The first one made use of an observation in [7] that allows to obtain asymptotics of the traces of convex functions of self-adjoint operators from the behaviour of their Riesz means. The result is then a consequence of the Karamata–Tauberian theorem (see e.g. [12, Theorem 10.3]) in a version that allows for logarithmic factors (see e.g. [11]). The second proof used a more direct approach in estimating the number of eigenvalues below a given value by Riesz means. Here, we present a proof that is in spirit very close to the latter argument, but emphasises the role of convexity. The proof method has been used in [1] in a similar context to our work, but has also been applied previously in a non-linear setting by Lieb and Simon [9], who give reference to Griffiths [4] for emphasising its use in mathematical physics.

Again, assume that W satisfies the assumptions of Proposition 1 and let λ_j^W denote the discrete eigenvalues of H_0+W. Note that for any $h>0$

$$\sum_{j\geq 1}(\lambda-\lambda_j^W)_+ \leq \sum_{j\geq 1}(\lambda+h-\lambda_j^W)_+ - hN^W(\lambda)$$

where $N^W(\lambda)$ denotes the number of eigenvalues λ_j^W below λ. As a consequence, we obtain the upper bound

$$N^W(\lambda) \leq \frac{1}{h}\left(\sum_{j\geq 1}(\lambda+h-\lambda_j^W)_+ - \sum_{j\geq 1}(\lambda-\lambda_j^W)_+\right). \tag{7}$$

To obtain a lower bound we observe that similarly

$$N^W(\lambda) \geq \frac{1}{h}\left(\sum_{j\geq 1}(\lambda-\lambda_j^W)_+ - \sum_{j\geq 1}(\lambda-h-\lambda_j^W)_+\right). \tag{8}$$

4.1. **Proof for hyperbolic cosine potential.** The result can be proved analogously to the case of polynomial potentials, for which details are presented below.

4.2. **Proof for polynomial potential.** Let λ_j be the eigenvalues of $H_0+x^{2N}+r(x)$. By Theorem 4 there are constants C_1, C_2 such that

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \frac{2}{\pi b}\left(\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda + C_1\lambda^{\frac{2N+1}{2N}}\right)$$

and

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \frac{2}{\pi b}\left(\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda + C_2\lambda^{\frac{2N+1}{2N}}\right)$$

for all sufficiently large λ. Inserting these bounds into (7) and noting that due to the convexity of the function $f(x)=x^{\frac{2N+1}{2N}}\log x$ for $x>1$ necessarily $f(\lambda+h)-f(\lambda)\leq hf'(\lambda+h)$, we obtain the upper bound

$$\frac{N(\lambda)}{\frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda} \leq \frac{(\lambda+h)^{\frac{1}{2N}}\log(\lambda+h)+\frac{2N}{2N+1}(\lambda+h)^{\frac{1}{2N}}}{\lambda^{\frac{1}{2N}}\log\lambda} + \frac{C_1(\lambda+h)^{\frac{2N+1}{2N}}-C_2\lambda^{\frac{2N+1}{2N}}}{h\lambda^{\frac{1}{2N}}\log\lambda}.$$

Choosing $h = (\rho - 1)\lambda$ with $\rho > 1$ and letting $\lambda \to \infty$ yields

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda} \leq \rho^{\frac{1}{2N}}$$

and since $\rho > 1$ was arbitrary

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda} \leq 1\,.$$

Similarly, we can use the convexity of $f(x) = x^{\frac{2N+1}{2N}}\log x$ for $x > 1$ to conclude that $f(\lambda) - f(\lambda - h) \geq hf'(\lambda - h)$ and thus obtain the upper bound

$$\frac{N(\lambda)}{\frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda} \geq \frac{(\lambda-h)^{\frac{1}{2N}}\log(\lambda-h) + \frac{2N}{2N+1}(\lambda-h)^{\frac{1}{2N}}}{\lambda^{\frac{1}{2N}}\log\lambda} + \frac{C_2\lambda^{\frac{2N+1}{2N}} - C_1(\lambda-h)^{\frac{2N+1}{2N}}}{h\lambda^{\frac{1}{2N}}\log\lambda}\,.$$

from (8). Choosing $h = (1 - \rho)\lambda$ with $\rho < 1$ and letting first $\lambda \to \infty$ and subsequently $\rho \to 1$ yields

$$\liminf_{\lambda\to\infty} \frac{N(\lambda)}{\frac{2}{\pi b}\lambda^{\frac{1}{2N}}\log\lambda} \geq 1\,.$$

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