# GENERALIZED HARDY INEQUALITY FOR THE MAGNETIC DIRICHLET FORMS

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*To Elliott H.Lieb, a man who loves inequalities, on the occasion of his birthday with our admiration and warmest wishes*

## 1. Introduction

The principal objective of this work is to obtain lower bounds for the magnetic form

$$h[u] = \int |(-i\nabla - \mathbf{a})u|^2 d\mathbf{x} \tag{1.1}$$

on $u \in \mathsf{C}_0^1(\mathbb{R}^d), d \geq 2$, with an appropriate vector-potential $\mathbf{a} \in \mathsf{L}^2(\mathbb{R}^d)$ having $d$ real-valued components. Probably the simplest way to generate lower bounds is to use the classical Hardy inequality

$$\int |\nabla u|^2 d\mathbf{x} \geq \frac{(d-2)^2}{4} \int \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x},$$

which holds for all dimensions $d \geq 3$ for $u \in \mathsf{C}_0^\infty(\mathbb{R}^d)$. It is worth pointing out that this bound remains valid if the $\nabla$-operator is replaced by its radial part. In view of the diamagnetic inequality (see (2.6)) the Hardy inequality implies the same bound for the form (1.1). In dimension $d = 2$ however, the Hardy inequality does not hold. Nevertheless, as was shown in [8], the magnetic field allows one to bound from below the angular part of the quadratic form (1.1) and to write a Hardy-type inequality for $d = 2$ with a constant $c$ depending strongly on the magnetic flux:

$$h[u] \geq c \int \frac{|u|^2}{1 + |\mathbf{x}|^2} d\mathbf{x}. \tag{1.2}$$

If the magnetic field $B = \operatorname{curl} \mathbf{a}$ is compactly supported and the total flux

$$\Phi = \frac{1}{2\pi} \int B d\mathbf{x}$$

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is integer, then $c = 0$. This inequality takes an especially elegant form in the case of the Aharonov-Bohm magnetic vector-potential:

$$h[u] \geq \min_{n\in\mathbb{Z}} |\Phi - n|^2 \int \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x},\ u \in \mathsf{C}_0^\infty(\mathbb{R}^2 \setminus \{0\}), \tag{1.3}$$

where $\Phi$ is the flux of the magnetic field. Clearly, the constant in this inequality vanishes if the flux is integer, that is if the field is gauge equivalent to the zero field. The proof of the estimates (1.2) and (1.3) is based on the observation that for a suitable choice of the gauge, the angular part of the quadratic form (1.1) is separated from zero if the flux $\Phi$ stays away from the set of integers. The result of [8] was extended to multiple Aharonov-Bohm fluxes in [1].

In the present paper we consider separately two cases: $d = 2$ and $d \geq 3$. To derive a meaningful estimate for $d = 2$ we exploit two elementary ideas. The first of them is the standard lower bound

$$h[u] \geq \int \pm B|u|^2 d\mathbf{x},$$

which holds with either of the signs $\pm$, see (2.13) below. The second ingredient is the Hardy inequality for domains with Lipshitz boundaries. Put together, these two ingredients yield a bound of the form

$$h[u] \geq c \int \tilde{B}|u|^2 d\mathbf{x},$$

with an *effective* magnetic field $\tilde{B}$, which coincides with $\pm B$ on its support, and decays outside the support as $\operatorname{dist}\{\mathbf{x},\ \operatorname{supp} B\}^{-2}$. The constant $c$ in the above estimate depends only on the support of $B$, see Theorem 2.1 for the precise statement. In contrast to (1.2) the constant $c$ is explicit and it does not depend on the flux.

In the case $d \geq 3$ the problem becomes more complicated, as the magnetic field $\mathbf{B} = d\mathbf{a}$ may now change its direction, see Subsection 2.3 for the precise definition of this notion. Assuming that the field never vanishes, under some extra conditions on the smoothness of $\mathbf{B}$ we prove the bound of the Sobolev type

$$h[u] \geq c \int |\mathbf{B}||u|^2 d\mathbf{x},$$

see Theorem 2.2. More detailed discussion of the result can be found in Subsection 2.4.

In contrast to the magnetic case there is an extensive literature on the non-magnetic Hardy and Sobolev inequalities. Various types of such non-magnetic inequalities could be found in books [15], [16], [12], [14] and in the review article [3]. Generalized Sobolev inequalities play an important role for Lieb-Thirring bounds and thus for the problem of stability of matter [10], [11], [13].

## 2. Main result and examples

2.1. **Magnetic form.** Let $\mathbf{a} = (a_1, a_2, ...a_d) \in \mathsf{L}^2_{\rm loc}(\mathbb{R}^d)$ be a real-valued vector function. Then the symmetric quadratic form

$$h[u] = \int |(\mathbf{D} - \mathbf{a})u|^2 d\mathbf{x},\ \mathbf{D} = -i\nabla,$$

is closable on $\mathsf{C}_0^\infty(\mathbb{R}^d)$. One can also define the magnetic operator $H = H_{\mathbf{a}}$ as the unique self-adjoint operator associated with the closure of the above form, but we do not need this fact in what follows. Assume that the two-form of the magnetic field $\mathbf{B} = d\mathbf{a}$ exists in the sense of distributions and it is a measurable on $\mathbb{R}^d$. We shell need the notation

$$b_{jk} = \partial_j a_k - \partial_k a_j, \quad k, j = 1, 2, \dots, d.$$

Since the two-form $\mathbf{B}$ is antisymmetric, it is fully determined by the components $b_{jk}$ with $j < k$. The number of these components is

$$\varkappa_d = d(d-1)/2.$$

We measure the strength of the field by the quantity

$$|\mathbf{B}| = \sqrt{\sum_{j<k} b_{jk}^2}.$$

In the two- and three-dimensional case this quantity coincides with the length of the magnetic field vector. If $d = 2$, then the only non-zero components of $\mathbf{B}$ are $b_{12}$ and $b_{21} = -b_{12}$.

In the next two subsections we state our results separately for two cases: $d = 2$ and $d \geq 3$. They have much in common but due to the simplicity of the magnetic field structure for $d = 2$, our results in this case are obtained under more general assumptions on the field than for $d \geq 3$. For both cases we need to introduce a positive continuous functions $\ell$ which plays the role of a slowly varying spatial scale reflecting variations of the magnetic field. We associate with the function $\ell$ the open ball

$$\mathcal{K}(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{y}| < \ell(\mathbf{x})\}.$$

The precise conditions on the function $\ell$ for $d = 2$ and $d \geq 3$ are slightly different and will be specified in each case separately.

2.2. **Results for** $d = 2$**.** Denote $B(\mathbf{x}) = b_{12}(\mathbf{x})$. The scale $\ell$ is assumed to satisfy the conditions

$$\ell \in \mathsf{C}^1(\mathbb{R}^2); \quad |\nabla \ell(\mathbf{x})| \leq 1, \quad \ell(\mathbf{x}) > 0,\ \forall \mathbf{x} \in \mathbb{R}^2. \tag{2.1}$$

To specify further conditions on $B$ we need to divide $\mathbb{R}^2$ into sets relevant to the strength of the field. For a (measurable) set $\mathcal{C} \subset \mathbb{R}^2$ define

$$\mathcal{C}^{\uparrow} = \cup_{\mathbf{x} \in \mathcal{C}} \mathcal{K}(\mathbf{x}). \tag{2.2}$$

With the field $B$ we associate two open sets $\Omega, \Lambda \subset \mathbb{R}^2$, such that $\Omega^\uparrow \subset \Lambda$ and $(\mathbb{R}^2 \setminus \Lambda)^\uparrow \cap \Omega^\uparrow = \emptyset$. The case $\Lambda = \mathbb{R}^2$ is not excluded. Let $\lambda_0 > 0$ be the lowest eigenvalue of the Laplace operator $-\Delta$ on the unit disk. with the Dirichlet boundary conditions. Denote

$$A_0 = \frac{5(2 + 4\sqrt{\lambda_0})}{\sqrt{2}}. \tag{2.3}$$

Assume that

$$|B(\mathbf{x})|\ell(\mathbf{x})^2 \geq 2A_0^2, \quad \text{a.a. } \mathbf{x} \in \Lambda. \tag{2.4}$$

The physical meaning of the sets $\Omega$, $\Lambda$ is that on $\Omega$ the field $B$ is "large", on $\mathbb{R}^2 \setminus \Lambda$ the field $B$ is negligibly small, and the set $\Lambda \setminus \Omega$ is a "transition zone".

Before stating the main result we need to introduce another important constant depending on $\Omega$. Suppose that the boundary of $\Omega$ is Lipshitz. Let $d(\mathbf{x})$ be the distance from $\mathbf{x} \in \mathbb{R}^2$ to $\Omega$. Then there exists a positive constant $\mu \leq 1/4$ such that for any $u \in \mathsf{H}_0^1(\Omega')$, $\Omega' = \mathbb{R}^2 \setminus \Omega$, one has the following Hardy inequality (see [9], [3] and [4]):

$$\int_{\Omega'} |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geq \mu \int_{\Omega'} \frac{|u(\mathbf{x})|^2}{d(\mathbf{x})^2} d\mathbf{x}. \tag{2.5}$$

If $\Omega'$ is a union of convex connected components, one has $\mu = 1/4$. In view of the diamagnetic inequality

$$\int |(\mathbf{D} - \mathbf{a})u|^2 d\mathbf{x} \geq \int |\nabla |u||^2 d\mathbf{x}, \ \forall u \in \mathsf{C}_0^1(\mathbb{R}^2), \tag{2.6}$$

we immediately infer from (2.5) that

$$\int_{\Omega'} |(\mathbf{D} - \mathbf{a})u(\mathbf{x})|^2 d\mathbf{x} \geq \mu \int_{\Omega'} \frac{|u(\mathbf{x})|^2}{d(\mathbf{x})^2} d\mathbf{x}, \ \forall u \in \mathsf{C}_0^1(\Omega'). \tag{2.7}$$

For other results connected with the inequality (2.5) and further references see, e.g. [2], [6].

**Theorem 2.1.** *Let $\Omega$ be an open set with Lipshitz boundary. Let the function $\ell$ be as specified in* (2.1)*, and let the field $B$ satisfy* (2.4)*. Suppose also that the field $B$ is either non-negative or non-positive a.a. $\mathbf{x} \in \mathbb{R}^2$. Then*

$$h[u] \geq \frac{\mu}{2} \int \frac{|u(\mathbf{x})|^2}{\ell(\mathbf{x})^2 + d(\mathbf{x})^2} d\mathbf{x}$$

*for all $u \in D[h]$.*

2.3. **Results for $d \geq 3$.** In this case our conditions on $\mathbf{B}$ are more restrictive. To state the precise conditions let us begin with the function $\ell$. We assume that

$$|\ell(\mathbf{x}) - \ell(\mathbf{y})| \leq \varrho|\mathbf{x} - \mathbf{y}|, \ 0 \leq \varrho < 1, \quad \ell(\mathbf{x}) > 0, \ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \tag{2.8}$$

Assume that for some $\Phi > 0$

$$|\mathbf{B}(\mathbf{x})|\ell(\mathbf{x})^2 \geq \Phi, \quad \text{a.a. } \mathbf{x} \in \mathbb{R}^d. \tag{2.9}$$

This assumption guarantees that the field $\mathbf{B}$ never vanishes. Denote by $\mathbf{n} = \{n_{jk}\}_{j,k=1}^d$ the matrix with the components

$$n_{jk}(\mathbf{x}) = \frac{b_{jk}(\mathbf{x})}{|\mathbf{B}(\mathbf{x})|},\ j,k = 1,2,\dots,d.$$

One may loosely call $\mathbf{n}$ the *direction matrix* for the field $\mathbf{B}$. Our second assumption on $\mathbf{B}$ is that for all $k,l = 1,2,\dots,d$ and $\mathbf{z} \in \mathbb{R}^d$

$$|\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})| \le \alpha, \forall \mathbf{x},\mathbf{y} \in \mathcal{K}(\mathbf{z}), \tag{2.10}$$

with some $0 \le \alpha \le \sqrt{\varkappa_d^{-1}}/4$. This assumption implies that the direction $\mathbf{n}$ of the field varies slowly with $\mathbf{x}$. Note that this condition is automatically fulfilled in the case $d = 2$ with $\alpha = 0$.

**Theorem 2.2.** *Let $d \ge 3$. Let the function $\ell$ be as specified in* (2.8)*, and let the field $\mathbf{B}$ be a continuous function satisfying* (2.9) *and* (2.10)*. Then for a sufficiently large $\Phi > 0$ in* (2.9) *one has*

$$h[u] \ge c \int |\mathbf{B}(\mathbf{x})||u(\mathbf{x})|^2 d\mathbf{x} \tag{2.11}$$

*for all $u \in D[h]$ with some positive constant $c$ depending on $\varrho$ and $\Phi$.*

Theorem 2.2 holds for the case $d = 2$ as well, but it gives nothing new compared to the inequality (2.13), see the discussion below.

Note that in contrast to Theorem 2.1 we do not specify the constant $c$ in the inequality (2.11), neither do we provide any precise estimates on the value of $\Phi$ sufficient for (2.11) to hold. In fact, as can be seen from the proof in Sect. 4, following the calculations carefully, one can always control the constants in all the estimates, but their values will hardly be optimal, and thus we do not go down this route.

2.4. **Discussion.** The simplest known source of lower bounds for the magnetic Schrödinger operator is the following representation for the quadratic form $h[u]$. Denote $\Pi_k = -i\partial_k - a_k$. Then

$$\|\Pi_k u\|^2 + \|\Pi_l u\|^2 = \|(\Pi_k \pm i\Pi_l)u\|^2 \pm (b_{kl}u, u),\ u \in \mathsf{C}_0^1(\mathbb{R}^d),$$

for any pair $k,l = 1,2,\dots,d$. This identity implies that

$$h[u] \ge \pm(b_{kl}u, u),\ \forall k,l = 1,2,\dots,d. \tag{2.12}$$

If one knows that for some $k,l$ the quantity $b_{kl}$ preserves its sign, and $c|\mathbf{B}| \le |b_{kl}|$, then the above inequality leads to the lower bound stated in Theorem 2.2. The above bound is especially useful in the case $d = 2$, in which it can be rewritten as follows:

$$h[u] \ge \pm(Bu, u),\ u \in \mathsf{C}_0^1(\mathbb{R}^2). \tag{2.13}$$

In fact, Theorem 2.1 trivially follows from this estimate and (2.4) if one assumes that $\Lambda = \mathbb{R}^2$. In this case, assuming, for instance, that $B > 0$ one uses (2.13) with the "+" sign, which leads, in view of (2.4) to the bound

$$h[u] \geq 2A_0^2 \int \frac{|u(\mathbf{x})|^2}{\ell(\mathbf{x})^2} d\mathbf{x}.$$

Since $2A_0^2 \geq 100$ and $\mu \leq 1/4$, this implies the sought lower bound. On the contrary, if $B \geq 0$ and the support of the field does not coincide with $\mathbb{R}^2$, then Theorem 2.1 yields a bound similar to (2.13), but with an *effective* magnetic field

$$\tilde{B}(\mathbf{x}) = \frac{1}{\ell(\mathbf{x})^2 + d(\mathbf{x})^2},$$

which, loosely speaking, coincides with $B$ inside the support, and decays away from it. It is important that this effective field does not vanish in contrast to $B$. Below we demonstrate how to use Theorem 2.1 for a few simple examples.

In the multi-dimensional situation the picture is different: the field $\mathbf{B}$ is allowed to change its direction. In these circumstances the estimate (2.12) is not very helpful as all the components $b_{kj}$ may change their signs. Theorem 2.2 is specifically designed to handle this situation. We need to assume however that $\mathbf{B}$ never vanishes.

A lower bound of a type similar to (2.11) was proved in [5]. Instead of the function $|\mathbf{B}|$ the inequality in [5] features a specific weight function, which coincides with $|\mathbf{B}|$ in the case of a polynomial magnetic field. Another instance when such an inequality is known to hold, is described in [17]. If the magnetic field is assumed to belong to a certain reverse Hölder class, then it is shown in [17] that $h[u] + \|u\|^2 \geq c(\ell^{-2}u, u)$ with some explicitly defined scale function $\ell$. Theorem 2.2 is close in the spirit to these results, but our proof is much more elementary, and it is based on a natural partition of unity associated with the scale function $\ell$.

2.5. **Examples.** If one wants to use Theorems 2.1 and 2.2, the first step is to make an appropriate choice of the function $\ell$ for a given magnetic field. Below we illustrate how it can be done in the case $d = 2$ for two special cases. The both examples are deliberately made strongly radially asymmetric in order to guarantee that the separation of variables is not applicable.

The first example is a *compactly supported magnetic field*. Let us denote by $D_R(x, y)$ the open disk in $\mathbb{R}^2$ of radius $R$, centered at $(x, y) \in \mathbb{R}^2$. Let $x_0 > R > 0$ and $\Lambda = D_R(x_0, 0) \cup D_R(-x_0, 0)$. Assume that

$$B(\mathbf{x}) = 0, \quad \mathbf{x} \notin \Lambda, \qquad B \geq B_0, \quad \mathbf{x} \in \Lambda \tag{2.14}$$

with some positive constant $B_0$. Let us define the function $\ell(\mathbf{x})$ by

$$\ell(\mathbf{x}) \equiv \ell_0 = \sqrt{2A_0^2/B_0}, \ \mathbf{x} \in \mathbb{R}^2. \tag{2.15}$$

Clearly, $\ell$ satisfies the conditions (2.1) and (2.4) on the set $\Lambda$. Moreover, $(\mathbb{R}^2\backslash\Lambda)^{\uparrow}\cap\Omega^{\uparrow} = \emptyset$. If $2\ell_0 < R$ then we can choose $\Omega$ as follows: $\Omega = D_{R-2\ell_0}(x_0, 0) \cup D_{R-2\ell_0}(-x_0, 0)$. Now Theorem 2.1 leads to the inequality

$$h[u] \geq \frac{\mu}{2} \int \frac{|u(\mathbf{x})|^2}{\ell_0^2 + d(\mathbf{x})^2} d\mathbf{x}, \quad u \in D[h], \tag{2.16}$$

where

$$d(x,y) = \begin{cases} 0, & \text{if } (x,y) \in \Omega, \\ \sqrt{(x-x_0)^2 + y^2} - (R - 2\ell_0), & \text{if } x \geq 0 \text{ and } (x,y) \notin \Omega, \\ \sqrt{(x+x_0)^2 + y^2} - (R - 2\ell_0), & \text{if } x \leq 0 \text{ and } (x,y) \notin \Omega. \end{cases}$$

Let us consider now an "opposite" example: a *magnetic fields with holes in its support*. Suppose as above that $R < x_0$. Assume again that $B$ satisfies (2.14) with the set

$$\Lambda = \mathbb{R} \setminus \overline{D_R(x_0, 0) \cup D_R(-x_0, 0)}$$

and define $\ell$ by (2.15). If $2\ell_0 < x_0 - R$, define $\Omega$ by

$$\Omega = \mathbb{R}^2 \setminus \overline{D_{R+2\ell_0}(x_0, 0) \cup D_{R+2\ell_0}(-x_0, 0)}$$

Then Theorem 2.1 yields again (2.16) with the distance function $d(\mathbf{x}) = d(x, y)$ given by

$$d(x,y) = \begin{cases} 0, & \text{if } (x,y) \in \Omega, \\ (R + 2\ell_0) - \sqrt{(x-x_0)^2 + y^2}, & \text{if } (x,y) \in D_{R+2\ell_0}(x_0, 0), \\ (R + 2\ell_0) - \sqrt{(x+x_0)^2 + y^2}, & \text{if } (x,y) \in D_{R+2\ell_0}(-x_0, 0). \end{cases}$$

Moreover, since $\Omega' = \mathbb{R}^2\backslash\Omega$ is a union of two convex sets (namely disks), one has $\mu = 1/4$, see the comment after formula (2.5).

Note that in both cases the effective field

$$\tilde{B}(\mathbf{x}) = \frac{1}{\ell_0^2 + d(\mathbf{x})^2}$$

in (2.16), shows the following behaviour in the strong field regime, that is when $B_0 \to \infty$. If $\mathbf{x} \in \Omega$, then $\tilde{B} \to \infty$ as well. For $\mathbf{x} \in \mathbb{R}^2 \setminus \Lambda$ the function $\tilde{B}$ behaves like $d(\mathbf{x})$, and thus, effectively it "does not feel" the magnetic field irrespectively of its strength. The set $\Lambda \setminus \Omega$, which consists of two rings of width $B_0^{-1/2}$, is a transition region.

Obviously, both examples can be generalized to any number of disks.

## 3. Proof of Theorem 2.1

3.1. **A partition of unity.** Let $\Omega, \Lambda \subset \mathbb{R}^2$ be the sets introduced in Subsect. 2.2, and let $\Omega^{\uparrow}$ be as defined in (2.2). As was previously mentioned, Theorem 2.1 trivially follows from (2.4) and (2.13) if $\Lambda = \mathbb{R}^2$. Henceforth we assume that $\mathbb{R}^2 \setminus \Lambda \neq \emptyset$.

Let $\Upsilon \in \mathsf{C}_0^1(\mathbb{R}^2)$ be a non-negative function such that $\Upsilon(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 1$ and $\int \Upsilon(\mathbf{x})^2 d\mathbf{x} = 1$. Denote

$$\lambda = \lambda(\Upsilon) = \int |\nabla\Upsilon(\mathbf{x})|^2 d\mathbf{x}. \tag{3.1}$$

Note that the lowest eigenvalue of $-\Delta$ on the unit disk is related to the number $\lambda$ in the following way:

$$\lambda_0 = \inf_{\Upsilon} \lambda. \tag{3.2}$$

This conclusion trivially follows from the variational principle.

**Lemma 3.1.** *Suppose that $\ell$ satisfies (2.1). Then the function*

$$\phi(\mathbf{x}) = \frac{1}{\ell(\mathbf{x})^2} \int_{\Omega^\uparrow} \Upsilon\left(\frac{\mathbf{x}-\mathbf{y}}{\ell(\mathbf{x})}\right)^2 d\mathbf{y}$$

*possesses the following properties:*

(i) $\phi \in \mathsf{C}^1(\mathbb{R}^2)$, *and* $|\nabla\phi(\mathbf{x})| \leq (2 + 4\sqrt{\lambda})\ell(\mathbf{x})^{-1}$;
(ii) $\phi(\mathbf{x}) = 1$ *for* $\mathbf{x} \in \Omega$, $\phi(\mathbf{x}) = 0$ *for* $\mathbf{x} \in \mathbb{R}^2 \setminus \Lambda$, *and* $0 \leq \phi(\mathbf{x}) \leq 1$.

*Proof.* The inclusion $\phi \in \mathsf{C}^1$ is obvious, since $\ell \in \mathsf{C}^1$. The estimate for $\nabla\phi$ is checked by a direct calculation:

$$\begin{aligned}
|\nabla\phi(\mathbf{x})| &\leq \frac{2|\nabla\ell(\mathbf{x})|}{\ell(\mathbf{x})^3} \int_{\Omega^\uparrow} \Upsilon(\,\cdot\,)^2 d\mathbf{y} + \frac{2}{\ell(\mathbf{x})^3} \int_{\Omega^\uparrow} |\Upsilon(\,\cdot\,)||\nabla\Upsilon(\,\cdot\,)| \left[1 + \frac{|\mathbf{x}-\mathbf{y}|}{\ell(\mathbf{x})}|\nabla\ell(\mathbf{x})|\right] d\mathbf{y} \\
&\leq \frac{2}{\ell(\mathbf{x})} + \frac{4}{\ell(\mathbf{x})^3} \int_{\Omega^\uparrow} |\Upsilon(\,\cdot\,)||\nabla\Upsilon(\,\cdot\,)| d\mathbf{y} \\
&\leq \frac{2}{\ell(\mathbf{x})} + \frac{4}{\ell(\mathbf{x})} \left[\int |\nabla\Upsilon(\mathbf{x})|^2 d\mathbf{x}\right]^{\frac{1}{2}} = \frac{2 + 4\sqrt{\lambda}}{\ell(\mathbf{x})}.
\end{aligned}$$

Here we have taken into account that $|\nabla\ell(\mathbf{x})| \leq 1$.

In view of the formula $\int \Upsilon(\mathbf{x})^2 d\mathbf{x} = 1$, we always have $\phi(\mathbf{x}) \leq 1$. Furthermore, if $\mathbf{x} \in \Omega$, then by definition $\mathcal{K}(\mathbf{x}) \subset \Omega^\uparrow$, and hence

$$\phi(\mathbf{x}) = \frac{1}{\ell(\mathbf{x})^2} \int_{\mathbb{R}^2} \Upsilon\left(\frac{\mathbf{x}-\mathbf{y}}{\ell(\mathbf{x})}\right)^2 d\mathbf{y} = 1,$$

as required. On the contrary, if $\mathbf{x} \in \mathbb{R}^2 \setminus \Lambda$, then by definition $\Upsilon\big((\mathbf{x}-\mathbf{y})\ell(\mathbf{x})^{-1}\big) = 0$ for all $\mathbf{y} \in \Omega^\uparrow$, and therefore $\phi(\mathbf{x}) = 0$. □

This Lemma allows one to introduce a convenient partition of unity:

**Lemma 3.2.** *Let the domains $\Omega$ and $\Lambda$ be as in Theorem 2.1, and let $\mathbb{R} \setminus \Lambda \neq \emptyset$. Then there exist two functions $\zeta, \eta \in \mathsf{C}^1(\mathbb{R}^2)$ such that*

(i) $0 \leq \zeta \leq 1, \quad 0 \leq \eta \leq 1$;
(ii) $\zeta(\mathbf{x}) = 1$ *for* $\mathbf{x} \in \Omega$, $\eta(\mathbf{x}) = 1$ *for* $\mathbf{x} \in \mathbb{R}^2 \setminus \Lambda$;
(iii) $\zeta^2 + \eta^2 = 1$;

(iv) $|\nabla\zeta| \le A\ell^{-1}$, $|\nabla\eta| \le A\ell^{-1}$ *with*

$$A = \frac{5(2+4\sqrt{\lambda})}{\sqrt{2}}. \tag{3.3}$$

*Proof.* Let $\phi$ be a function constructed in Lemma 3.1, and let $\psi = 1-\phi$. It is straightforward to check that $\phi^2+\psi^2 = 2(\phi-1/2)^2+1/2 \ge 1/2$. Define

$$\zeta = \frac{\phi}{\sqrt{\phi^2+\psi^2}},\ \eta = \frac{\psi}{\sqrt{\phi^2+\psi^2}}.$$

These functions, obviously satisfy properties (i), (ii), (iii). To prove (iv) note that

$$|\nabla\zeta| \le \frac{5}{2\sqrt{\phi^2+\psi^2}}|\nabla\phi| \le \frac{5}{\sqrt{2}}|\nabla\phi|,$$

and a similar bound holds for $\nabla\eta$. Now the required estimate follows from Lemma 3.1. □

3.2. **Proof of Theorem 2.1.** Suppose that the conditions of Theorem 2.1 are fulfilled. Our next step is to split the magnetic form $h[u]$ into two parts that will be estimated in two different ways. Let $\zeta, \eta$ be the functions from Lemma 3.2. Since $\zeta^2+\eta^2 = 1$, we have for any $u \in \mathsf{C}_0^1(\mathbb{R}^2)$:

$$\begin{aligned} h[u] &= \int |\zeta(\mathbf{D}-\mathbf{a})u|^2 d\mathbf{x} + \int |\eta(\mathbf{D}-\mathbf{a})u|^2 d\mathbf{x} \\ &= h[\zeta u] + h[\eta u] - \int (|\nabla\zeta|^2 + |\nabla\eta|^2)|u|^2 d\mathbf{x}. \end{aligned}$$

We use the following decomposition:

$$h[u] = \frac{1}{2}\left[h[u] - \int (|\nabla\zeta|^2+|\nabla\eta|^2)|u|^2 d\mathbf{x}\right] + \frac{1}{2}h[\zeta u] + \frac{1}{2}h[\eta u]. \tag{3.4}$$

Since $B$ does not change sign, it follows from (2.13) that $h[u] \ge (|B|u,u)$. Let us estimate from below the first term in the r.h.s. of (3.4), remembering that $\nabla\zeta$ and $\nabla\eta$ are supported on the set $\Lambda$:

$$h[u] - \int (|\nabla\zeta|^2+|\nabla\eta|^2)|u|^2 d\mathbf{x} \ge \int_\Lambda \left[|B| - (|\nabla\zeta|^2+|\nabla\eta|^2)\right]|u|^2 d\mathbf{x}.$$

In view of the condition (2.4) and of the fact that $|\nabla\zeta|^2+|\nabla\eta|^2 \le 2A^2\ell^{-2}$, the r.h.s. is bounded from below by $-\nu E(u)$ with

$$E(u) = \int_\Lambda \frac{1}{\ell(\mathbf{x})^2}|u(\mathbf{x})|^2 d\mathbf{x}, \quad \nu = 2(A^2 - A_0^2) \ge 0.$$

Let us estimate the remaining two terms in (3.4). For the term with $\zeta$ use (2.4) again, keeping in mind that $\zeta$ is supported on $\Lambda$:

$$h[\zeta u] \geq \int |B(\mathbf{x})|\zeta(\mathbf{x})^2|u(\mathbf{x})|^2 d\mathbf{x} \geq 2A_0^2 \int \frac{1}{\ell(\mathbf{x})^2}\zeta(\mathbf{x})^2|u|^2 d\mathbf{x}.$$

For the term with $\eta$ use Hardy's inequality (2.7):

$$h[\eta u] \geq \mu \int \eta(\mathbf{x})^2 \frac{|u(\mathbf{x})|^2}{d(\mathbf{x})^2} d\mathbf{x},\ \ d(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \Omega).$$

Collecting all the estimates we now obtain the following lower bound:

$$2h[u] \geq 2A_0^2 \int \zeta^2 \frac{|u|^2}{\ell^2} d\mathbf{x} + \mu \int \eta^2 \frac{|u|^2}{d^2} d\mathbf{x} - 2\nu E(u).$$

Since $2A_0^2 \geq 100$ and $\mu \leq 1/4$, one can write

$$2h[u] \geq \mu \int \frac{|u|^2}{d^2 + \ell^2} d\mathbf{x} - 2\nu E(u).$$

Neither the r.h.s. nor l.h.s. depend on the function $\Upsilon$. Therefore we can take the sup of both sides over all admissible $\Upsilon$. In view of definitions (2.3)and (3.3), the equality (3.2) yields $\sup_\Upsilon(-\nu) = \inf_\Upsilon \nu = 0$. This leads to the required bound from below and thus completes the proof of Theorem 2.1.

## 4. Proof of Theorem 2.2

Suppose that the condition of Theorem 2.2 are fulfilled, and in particular, the function $\ell$ satisfies (2.8).

### 4.1. **Partition of unity.**

The key stone of the proof is the following partition of unity associated with the scale function $\ell(\mathbf{x})$.

**Lemma 4.1.** *Let $\ell(\mathbf{x})$ (resp. $\ell(x)$) be a continuous function satisfying* (2.8). *Then there exists a set of points $\mathbf{x}_j \in \mathbb{R}^d$, $j \in \mathbb{N}$ such that the open balls $\mathcal{K}_j = \mathcal{K}(\mathbf{x}_j)$ form a covering of $\mathbb{R}^d$ with the finite intersection property (i.e. each ball $\mathcal{K}_j$ intersects with no more than $\tilde{N} = \tilde{N}(\varrho) < \infty$ other balls). Moreover, there exists a set of non-negative functions $\phi_j \in \mathsf{C}_0^\infty(K_j)$, $j \in \mathbb{N}$, such that*

$$\sum_j \phi_j^2 = 1, \tag{4.1}$$

*and*

$$|\partial^m \phi_j| \leq C_m \ell^{-|m|},\ \ \forall m, \tag{4.2}$$

*uniformly in $j$.*

Emphasize that the square in (4.1) will be convenient for us, though the common definition of the partition of unity requires $\sum_j \phi_j = 1$. Proof of this Lemma is analogous to that of Theorem 1.4.10 from [7] and we do not reproduce it here.

Let us rephrase the finite intersection property for balls $\mathcal{K}_j$ as follows. Denote

$$\mathfrak{m}_j = \{k \in \mathbb{N} : \mathcal{K}_j \cap \mathcal{K}_k \neq \emptyset\}.$$

Then

$$\operatorname{card} \mathfrak{m}_j \le N(\varrho) := \tilde{N}(\varrho) + 1,$$

with the number $\tilde{N}(\varrho)$ defined in Lemma 4.1.

4.2. **Partition of the magnetic form: proof of Theorem 2.2.** Let us use the partition of unity constructed in Lemma 4.1. Let $u \in \mathsf{C}_0^1(\mathbb{R}^d)$. Then a simple calculation, similar to that in the proof of Theorem 2.1, shows that

$$h[u] = \sum_k h[\phi_k u] - \sum_k \int |\nabla \phi_k|^2 |u|^2 d\mathbf{x}. \tag{4.3}$$

Estimate the first term in the r.h.s. as follows:

$$N \sum_k h[\phi_k u] \ge \sum_k \sum_{l \in \mathfrak{m}_k} h[\phi_l u], \ \ N = N(\varrho).$$

Let us fix $k \in \mathbb{N}$. Let $j, l \in [1, d]$ be a pair of integers such that $|b_{jl}(\mathbf{x}_k)| \ge \sqrt{\varkappa_d^{-1}} |\mathbf{B}(\mathbf{x}_k)|$. Such a pair always exists. Assume without loss of generality that $b_{jl} > 0$. Then, in view of (2.10)

$$b_{jl}(\mathbf{x}) \ge \frac{3}{4\sqrt{\varkappa_d}} |\mathbf{B}(\mathbf{x})|, \ \ \mathbf{x} \in \mathcal{K}_k.$$

Using (2.10) again we obtain

$$b_{jl}(\mathbf{x}) \ge \frac{1}{2\sqrt{\varkappa_d}} |\mathbf{B}(\mathbf{x})|, \ \ \mathbf{x} \in \cup_{s \in \mathfrak{m}_k} \mathcal{K}_s.$$

Estimate from below using (2.12):

$$\sum_{s \in \mathfrak{m}_k} h[\phi_s u] \ge \int b_{jl} \sum_{s \in \mathfrak{m}_k} \phi_s^2 |u|^2 d\mathbf{x} \ge \frac{1}{2\sqrt{\varkappa_d}} \int |\mathbf{B}| \sum_{s \in \mathfrak{m}_k} \phi_s^2 |u|^2 d\mathbf{x}.$$

The last integral is bounded from below by

$$\frac{1}{2\sqrt{\varkappa_d}} \int_{\mathcal{K}_k} |\mathbf{B}| |u|^2 d\mathbf{x}.$$

Here we have used the fact that $\sum_{s \in \mathfrak{m}_k} \phi_s(\mathbf{x})^2 = 1$ for all $\mathbf{x} \in \mathcal{K}_k$, by definition of $\mathfrak{m}_k$. Consequently

$$\sum_k h[\phi_k u] \ge \frac{1}{N} \sum_k \sum_{s \in \mathfrak{m}_k} h[\phi_s u] \ge \frac{1}{2N\sqrt{\varkappa_d}} \sum_k \int_{\mathcal{K}_k} |\mathbf{B}| \ |u|^2 d\mathbf{x}. \tag{4.4}$$

Let us estimate $h[u]$ from below using (4.3) and (4.4):

$$h[u] \geq \sum_k \int_{\mathcal{K}_k} \left[\frac{1}{2N\sqrt{\varkappa_d}}|\mathbf{B}| - |\nabla\phi_k|^2\right] |u|^2 d\mathbf{x}.$$

Here we have used the fact that $\nabla\phi_k$ is supported on $\mathcal{K}_k$. According to (2.9) and (4.2) we have

$$|\nabla\phi_k|^2 \leq c^2\ell^{-2} \leq c^2\Phi^{-1}|\mathbf{B}|,$$

so that

$$h[u] \geq \left[\frac{1}{2N\sqrt{\varkappa_d}} - c^2\Phi^{-1}\right] \sum_k \int_{\mathcal{K}_k} |\mathbf{B}|\ |u|^2 d\mathbf{x}.$$

Assuming that $\Phi$ is sufficiently large, so that the factor before the integral is positive, we get

$$h[u] \geq c\int |\mathbf{B}|\ |u|^2 d\mathbf{x},\ u \in \mathsf{C}_0^1(\mathbb{R}^d).$$

This is the required bound. Proof of Theorem 2.2 is now complete.

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