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WEIGHTED CLR TYPE BOUNDS IN TWO DIMENSIONS

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ABSTRACT. We derive weighted versions of the Cwikel–Lieb–Rozenblum inequality for the Schrödinger operator in two dimensions with a nontrivial Aharonov–Bohm magnetic field. Our bounds capture the optimal dependence on the flux and we identify a class of long-range potentials that saturate our bounds in the strong coupling limit. We also extend our analysis to the two-dimensional Schrödinger operator acting on antisymmetric functions and obtain similar results.

1. INTRODUCTION AND MAIN RESULTS

The celebrated Cwikel–Lieb–Rozenblum (CLR) inequality states that the number $N(-\Delta - V)$ of negative eigenvalues, including multiplicity, of a Schrödinger operator $-\Delta - V$ in $L^2(\mathbb{R}^d)$ in dimension $d \geqslant 3$ is bounded by

$$N(-\Delta - V) \lesssim_d \int_{\mathbb{R}^d} V(x)_+^{d/2} \,\mathrm{d}x \tag{1}$$

where the implied constant is independent of V. Here and throughout we take $a_\pm := \max(0, \pm a)$ and use a subscript on $\lesssim$ to specify the variables on which the implied constant depends. The inequality is due to M. Cwikel [7], E. Lieb [24] and G. Rozenbljum [27]. For further proofs and background we direct the reader to [12]. The bound is saturated in the strong coupling limit, that is where V is replaced with λV and $\lambda \to \infty$, since by Weyl's asymptotics,

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N(-\Delta - \lambda V) = \frac{\omega_d}{(2\pi)^d} \int_{\mathbb{R}^d} V(x)_+^{d/2} \,\mathrm{d}x, \tag{2}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. One of the uses of (1) is to extend this asymptotic behavior, which is originally established for instance for continuous V of compact support, to all V with $V_+ \in L^{d/2}(\mathbb{R}^d)$. Concerning the repulsive part one only needs to assume $V_- \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ [10].

Building on earlier work for radial potentials by V. Glaser, H. Grosse and A. Martin [13], the CLR inequality was generalised by Y. Egorov and V. Kondratiev in [8]

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to include the weighted bounds

$$N(-\Delta - V) \lesssim_{d,\alpha} \int_{\mathbb{R}^d} V(x)_+^{(d+\alpha)/2} |x|^\alpha \, \mathrm{d}x, \tag{3}$$

which hold in dimensions $d \geqslant 3$ for any $\alpha > 0$. In [5], M. Birman and M. Solomyak showed that the strong L^p norm appearing on the right in (3) can be replaced by a weak norm, namely

$$N(-\Delta - V) \lesssim_{d,\alpha} \sup_{t>0} t^{(d+\alpha)/2} \int_{|x|^2 V(x)_+ > t} \frac{\mathrm{d}x}{|x|^d}, \tag{4}$$

which is valid, again, in dimensions $d \geqslant 3$ with $\alpha > 0$. Note that the bounds (3) and (4) are homogeneous with respect to V of degree $(d+\alpha)/2 > d/2$, in contrast to the homogeneity $d/2$ of (1). The latter homogeneity is consistent with (2). Nevertheless, as shown by M. Birman and M. Solomyak [2], the asymptotic order of growth $(d+\alpha)/2$ in (4) can be saturated in the strong coupling limit for a class of potentials with particular long range behaviour. Namely, if $V_+ \in L^{d/2}_{\mathrm{loc}}(\mathbb{R}^d)$ satisfies

$$V(x) = |x|^{-2} |\ln|x||^{-1/p} \left(1 + o(1)\right) \text{ as } |x| \to \infty \tag{5}$$

for some $p > d/2$, then one can show that

$$\lim_{\lambda\to\infty} \lambda^{-p} N(-\Delta - \lambda V) \text{ exists and is finite,}$$

while for $\alpha > 0$ with $p = (d+\alpha)/2$,

$$\lim_{\lambda\to\infty} \lambda^{-p} \sup_{t>0} t^{(d+\alpha)/2} \int_{\lambda|x|^2 V(x)_+ > t} \frac{\mathrm{d}x}{|x|^d} = \sup_{t>0} t^p \int_{|x|^2 V(x)_+ > t} \frac{\mathrm{d}x}{|x|^d} \in (0,\infty).$$

All the results discussed so far are restricted to the case of dimensions $d \geqslant 3$ and most of their direct analogues in dimensions $d = 2$ fail. For instance, none of the direct analogues of (1), (3) and (4) hold. Moreover, there are examples of $V \in L^1(\mathbb{R}^2)$ with $V \geqslant 0$ for which either the limit on the left side of (2) is infinite or it is finite but different from the right side, see [4]. Recently, there has been a lot of activity in proving bounds on $N(-\Delta - V)$ in $d = 2$ and in giving necessary and sufficient conditions for either the bound $\lim_{\lambda\to\infty} \lambda^{-1} N(-\Delta - \lambda V) < \infty$ or the validity of (2). A sample of references for this development is [14, 16, 20, 21, 26, 29]. An earlier fundamental paper is due to M. Solomyak [30]; see also [11].

In this paper we are concerned with bounds on the number of negative eigenvalues of two-dimensional Schrödinger operators in the presence of an Aharonov–Bohm magnetic field. We will see that when this field is nontrivial, one obtains inequalities that are analogous to those discussed above for Schrödinger operators in dimensions $d \geqslant 3$ and see that the difficulties of the two-dimensional case mostly disappear. We will also consider the case of the non-magnetic Schrödinger operator restricted to antisymmetric functions and see that this case is similar to that of an Aharonov–Bohm magnetic field.

Our results support the heuristics that the different behaviour in dimensions $d \geqslant 3$ and in $d = 2$ comes from a spectral instability of the two-dimensional Laplacian near energy zero and that this instability can be removed by additional repulsion, either in the form of a magnetic field or the presence of symmetries. For other instances of this principle see [19, 22].

To be more specific, let

$$\mathbf{A}(x) = |x|^{-2}(x_2, -x_1) \qquad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2$$

and for $\Phi \in \mathbb{R}$ let

$$D_\Phi = -i\nabla + \Phi\mathbf{A}\,.$$

We consider the magnetic Schrödinger operators

$$D_\Phi^2 - V \text{ in } L^2(\mathbb{R}^2)\,.$$

As discussed in the next section, under suitable conditions on V this operator can be realized as a self-adjoint operator via the closure of the corresponding quadratic form on $C_0^\infty(\mathbb{R}^2\backslash\{0\})$. When $\Phi \in \mathbb{Z}$, the magnetic potential can be gauged away and the operator is unitarily equivalent to $-\Delta + V$. Therefore, in the following we will concentrate on the case $\Phi \in \mathbb{R}\backslash\mathbb{Z}$.

An analogue of the CLR inequality (1) was shown by A. Balinsky, W. Evans and R. Lewis [1], namely,

$$N(D_\Phi^2 - V) \lesssim_\Phi \int_0^\infty \sup_{\omega\in\mathbb{S}} V(r\omega)_+ r\,\mathrm{d}r. \tag{6}$$

More recently it was deduced in [22] that when V_+ is radially non-increasing one can replace the supremum over angles in the right side of (6) with an integral, that is,

$$N(D_\Phi^2 - V) \lesssim_\Phi \int_{\mathbb{R}^2} V(x)_+\,\mathrm{d}x. \tag{7}$$

However, it is known [1] that this replacement cannot be made for general $V \in L^1(\mathbb{R}^2)$.

Our main result is the following magnetic version of (3).

Theorem 1. *Let $\Phi \in \mathbb{R}\backslash\mathbb{Z}$ and $\alpha > 0$. Then there is a constant $C_{\Phi,\alpha} < \infty$ such that*

$$N(D_\Phi^2 - V) \leqslant C_{\Phi,\alpha} \int_{\mathbb{R}^2} V(x)_+^{1+\alpha/2} |x|^\alpha\,\mathrm{d}x \tag{8}$$

for all $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for which the right side is finite. Moreover, the optimal constant in this inequality satisfies

$$C_{\Phi,\alpha} \sim_\alpha d(\Phi)^{-1-\alpha} \tag{9}$$

with $d(\Phi) := \min_{k\in\mathbb{Z}} |\Phi - k|$.

In fact, our proof yields the explicit upper bound

$$C_{\Phi,\alpha} \leqslant \frac{\Gamma((1+\alpha)/2)}{4\pi^{3/2}\Gamma(1+\alpha/2)} \sum_{n\in\mathbb{Z}} |n-\Phi|^{-1-\alpha}\,. \tag{10}$$

From this bound we immediately obtain the upper bound $C_{\Phi,\alpha} \lesssim_\alpha d(\Phi)^{-1-\alpha}$ in (9). In the proof of Theorem 1 we will show that this bound is sharp, thereby obtaining the precise divergence of the constant as the flux Φ approaches an integer value.

We complement Theorem 1 with a variant of this bound with a weak norm.

Corollary 2. *Let $\Phi \in \mathbb{R}\backslash\mathbb{Z}$ and $\alpha > 0$. Then there is a constant $C'_{\Phi,\alpha} < \infty$ such that*

$$N(D_\Phi^2 - V) \leqslant C'_{\Phi,\alpha} \sup_{t>0} t^{1+\alpha/2} \int_{|x|^2 V(x)_+ > t} \frac{\mathrm{d}x}{|x|^2} \tag{11}$$

for all $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for which the right side is finite. Moreover, the constant can be chosen to satisfy

$$C'_{\Phi,\alpha} \sim_\alpha d(\Phi)^{-1-\alpha} . \tag{12}$$

Since

$$\sup_{t>0} t^{1+\alpha/2} \int_{|x|^2 V(x)_+ > t} \frac{\mathrm{d}x}{|x|^2} \leqslant \int_{\mathbb{R}^2} V(x)_+^{1+\alpha/2} |x|^\alpha \, \mathrm{d}x ,$$

the bound (8) follows from (11) and for the sharp constants we find

$$C_{\Phi,\alpha} \leqslant C'_{\Phi,\alpha} . \tag{13}$$

We will argue differently, however, and deduce Corollary 2 from Theorem 1. To do this, we use an interpolation argument in the spirit of one of M. Birman and M. Solomyak [5].

In further likeness to the situation for $-\Delta - V$ in dimensions $d \geqslant 3$, we derive examples of potentials with the same long-range behaviour (5) that saturate the weak inequality (11) in the strong coupling limit. We refer to Section 4 for the details. There we will show, in particular,

$$C'_{\Phi,\alpha} \geqslant \frac{\Gamma((1+\alpha)/2)}{4\pi^{3/2}\Gamma(1+\alpha/2)} \sum_{n\in\mathbb{Z}} |n-\Phi|^{-1-\alpha} , \tag{14}$$

which should be compared with (10). Of course, these two bounds are consistent with (13).

We note that in [17], Kovařík considers the two-dimensional Schrödinger operator with a general, nontrivial, magnetic potential $A \in L^2_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$. Under minimal assumptions on A it is determined that for any $\alpha > 0$

$$N((-\mathrm{i}\nabla + A)^2 - V) \lesssim_{A,\alpha} \int_{\mathbb{R}^2} V(x)_+^{1+\alpha/2} (1+|x|)^\alpha \, \mathrm{d}x .$$

This bound is similar to ours, but does not include it, for our $A = \Phi\mathbf{A}$ is not locally integrable around the origin. If one could adapt the argument to include our A, then a scaling argument could be used to remove the extra 1 in the weight and one would obtain a bound of the form (8). Our analysis has the merit of yielding tighter constants through a more direct proof, as well as establishing the optimal constant-flux relationship.

Next, we describe our results for two-dimensional Schrödinger operators acting on antisymmetric functions. For functions V on $\mathbb{R}^2$ that are symmetric in the sense that $V(x_1,x_2) = V(x_2,x_1)$ for almost every $x \in \mathbb{R}^2$ we can consider the operator $-\Delta - V$ in $L^2(\mathbb{R}^2)$ restricted to antisymmetric functions, that is, in the Hilbert space

$$L^2_{\mathbf{as}}(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) : u(x_1,x_2) = -u(x_2,x_1) \text{ for almost every } x \in \mathbb{R}^2\} .$$

We denote the resulting operator by $-\Delta_{\mathbf{as}} - V$. Under the assumption that V is radially non-increasing, a corresponding version of the CLR inequality for this operator was found in [22], namely

$$N(-\Delta_{\mathbf{as}} - V) \lesssim \int_{\mathbb{R}^2} V(x)_+ \, \mathrm{d}x.$$

However, this inequality does not hold for general V, as noted in [22, Remark 1].

Our second pair of main results are strong and weak weighted CLR bounds for $-\Delta_{\mathbf{as}} - V$, analogous to the bounds we derived for the magnetic operator.

Theorem 3. *Let $\alpha > 0$, then there is a constant $C_\alpha < \infty$ such that*

$$N(-\Delta_{\mathbf{as}} - V) \leqslant C_\alpha \int_{\mathbb{R}^2} V(x)_+^{1+\alpha/2} |x|^\alpha \, \mathrm{d}x \tag{15}$$

for all symmetric $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for which the right side is finite.

In fact, our proof yields the explicit upper bound

$$C_\alpha \leqslant \frac{\Gamma((1+\alpha)/2)}{2\pi^{3/2}\Gamma(1+\alpha/2)} \zeta(1+\alpha)\,, \tag{16}$$

where ζ is the Riemann zeta function.

Corollary 4. *Let $\alpha > 0$, then there is a constant $C'_\alpha < \infty$ such that*

$$N(-\Delta_{\mathbf{as}} - V) \leqslant C'_\alpha \sup_{t>0} t^{1+\alpha/2} \int_{|x|^2 V(x)_+ > t} \frac{\mathrm{d}x}{|x|^2} \tag{17}$$

for all symmetric $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for which the right side is finite.

Again, for long-range potentials of the form (5) the bound in the corollary can be saturated in the strong coupling limit and one obtains the lower bound

$$C'_\alpha \geqslant \frac{\Gamma((1+\alpha)/2)}{2\pi^{3/2}\Gamma(1+\alpha/2)} \zeta(1+\alpha)\,.$$

Our plan for the paper is as follows: In Section 2 we present the proof of Theorems 1 and 3. In Section 3 we derive the weak forms of the inequalities above. Finally, in Section 4 we will show that these bounds are saturated in the strong coupling limit by potentials with long range behaviour (5).

2. Proof of theorems 1 and 3

2.1. **The Aharonov–Bohm operator.** We begin by showing that the operators $D_\Phi^2 - V$ are well-defined in quadratic form sense when $\Phi \in \mathbb{R}\backslash\mathbb{Z}$ and V is such that the right side in either Theorem 1 or Corollary 2 is finite. The main ingredient in this argument is the magnetic Hardy–Sobolev inequality

$$\int_{\mathbb{R}^2} |D_\Phi u|^2 \, \mathrm{d}x \geqslant S_{\Phi,q} \left(\int_{\mathbb{R}^2} \frac{|u|^q}{|x|^2} \, \mathrm{d}x \right)^{2/q} \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^2\backslash\{0\})\,, \tag{18}$$

with $S_{\Phi,q} > 0$ provided that $q \in [2, \infty)$. A proof of this inequality can be found in [6, Section 3.1, Step 1] based on the diamagnetic inequality and a special case of the Caffarelli–Kohn–Nirenberg inequality for scalar functions. Alternatively, one can deduce this inequality using the method of [8]. In the special case $q = 2$ inequality (18) with sharp constant is due to [23] and reads

$$\int_{\mathbb{R}^2} |D_\Phi u|^2 \, \mathrm{d}x \geqslant d(\Phi)^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, \mathrm{d}x\,. \tag{19}$$

Some results about the sharp constant in (18) for $q > 2$ can be found in [6].

Let us show how to use (18) to define the operator $D_\Phi^2 - V$. We combine (18) with Hölder's inequality to obtain for $u \in C_c^\infty(\mathbb{R}^2\backslash\{0\})$

$$\begin{aligned}\int_{\mathbb{R}^2} V_+ |u|^2 \, \mathrm{d}x &\leqslant \left(\int_{\mathbb{R}^2} V_+^{1+\alpha/2} |x|^\alpha \, \mathrm{d}x \right)^{1/(1+\alpha/2)} \left(\int_{\mathbb{R}^2} \frac{|u|^q}{|x|^2} \, \mathrm{d}x \right)^{2/q} \\ &\leqslant S_{\Phi,q}^{-1} \left(\int_{\mathbb{R}^2} V_+^{1+\alpha/2} |x|^\alpha \, \mathrm{d}x \right)^{1/(1+\alpha/2)} \int_{\mathbb{R}^2} |D_\Phi u|^2 \, \mathrm{d}x\,,\end{aligned} \tag{20}$$

where q and α are related by $1/(1+\alpha/2)+2/q=1$. The assumption $q<\infty$ is equivalent to $\alpha>0$.

Now given $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that the integral in Theorem 1 is finite and given $\varepsilon>0$, we decompose $V_+ = V_1 + V_2$ with $V_2 \in L^\infty(\mathbb{R}^2)$ and $V_1 \geqslant 0$ satisfying

$$\int_{\mathbb{R}^2} V_1^{1+\alpha/2} |x|^\alpha \,\mathrm{d}x \leqslant \varepsilon .$$

Applying (20) with V_1 we find that V_+ is form-bounded with respect to $D_\Phi^2 + V_-$ with relative form bound zero. This allows us to define $D_\Phi^2 - V$ as a selfadjoint, lower semibounded operator in $L^2(\mathbb{R}^2)$ with form core $C_c^\infty(\mathbb{R}^2 \backslash \{0\})$.

Meanwhile, let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ be given such that the integral in Corollary 2 is finite and let $\varepsilon>0$. We choose $\tilde q \in (q,\infty)$ and define $\tilde\alpha>0$ by $1/(1+\tilde\alpha/2)+2/\tilde q = 1$. We can decompose $V_+ = V_1 + V_2$ with $\||x|^2 V_2\|_{L^\infty(\mathbb{R}^2)} \leqslant \varepsilon$ and $V_1 \geqslant 0$ satisfying

$$\int_{\mathbb{R}^2} V_1^{1+\tilde\alpha/2} |x|^{\tilde\alpha} \,\mathrm{d}x < \infty .$$

(Indeed, we can simply take $V_1 = |x|^{-2}(|x|^2 V_+ - \varepsilon)_+$ and apply the layer cake representation; refer to Section 3.) Proceeding as before to control the V_1 piece and using (19) to control the V_2 piece, we find again that V_+ is form-bounded with respect to $D_\Phi^2 + V_-$ with relative bound zero and, consequently, that $D_\Phi^2 - V$ is well-defined.

Next, we recall that the operators $D_\Phi^2 - V$ and $D_{\Phi-k}^2 - V$ are unitarily equivalent for $k \in \mathbb{Z}$ and that the operators $D_\Phi^2 - V$ and $D_{-\Phi}^2 - V$ are antiunitarily equivalent; see, e.g., [6, Subsection 2.1]. Thus, in what follows we can restrict ourselves to the case $\Phi \in (0,1/2]$.

We are now ready to present the proof of the weighted CLR bound for $D_\Phi^2 - V$.

Proof of Theorem 1. Fix $\alpha>0$ and let $V_+|x|^2 \in L^{1+\alpha/2}(\mathbb{R}^2; \mathrm{d}x/|x|^2)$. As explained above, we may assume $\Phi \in (0,1/2]$. Moreover, by the variational principle, we may assume $V \geqslant 0$. According to (20) the Birman–Schwinger operator $V^{1/2}(D_\Phi^2)^{-1}V^{1/2}$ is well-defined and bounded. Changing to polar coordinates and logarithmic variables, this operator becomes $\widetilde V_+^{1/2}(-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2)^{-1}\widetilde V_+^{1/2}$ in $L^2(\mathbb{R}\times\mathbb{S}^1)$, where

$$\widetilde V(t,\theta) = e^{2t} V(e^t\cos\theta, e^t\sin\theta) .$$

Applying the Birman–Schwinger principle (see, e.g., [12, Subsection 4.3.3]) and the Lieb–Thirring inequality (see [25] and also [12, Theorem 4.59]) we obtain that for $p = 1+\alpha/2 > 1$

$$\begin{aligned} N(D_\Phi^2 - V) &\leqslant n_+(1, \widetilde V_+^{1/2}(-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2)^{-1}\widetilde V_+^{1/2}) \\ &\leqslant \operatorname{Tr}(\widetilde V_+^{1/2}(-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2)^{-1}\widetilde V_+^{1/2})^p \\ &\leqslant \operatorname{Tr}(\widetilde V_+^{p/2}(-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2)^{-p}\widetilde V_+^{p/2}). \end{aligned} \tag{21}$$

To compute the trace we need to find the integral kernel of the operator $(-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2)^{-p}$, which we denote by $G_{\Phi,p}(t,\theta;\tau,\vartheta)$. We note that $(-\partial_t^2 + (i\partial_\theta - \Phi)^2)$ in $L^2(\mathbb{R}\times\mathbb{S})$ is unitarily equivalent, via a continuous and a discrete Fourier transform, to multiplication by $\xi^2 + (n-\Phi)^2$ in $L^2(\mathbb{R}) \times \ell^2(\mathbb{Z})$. Thus,

$$G_{\Phi,p}(t,\theta;\tau,\vartheta) = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{e^{\mathrm{i}n(\theta-\vartheta)} e^{\mathrm{i}\xi(t-\tau)}}{(\xi^2 + (n-\Phi)^2)^p} \,\mathrm{d}\xi .$$

Given that $\Phi \in (0, 1/2]$ and $p > 1$ the above sum converges. Moreover, $g_{\Phi,p} := G_{\Phi,p}(t, \theta; t, \theta)$ is independent of t and θ and we compute that

$$g_{\Phi,p} = \frac{\Gamma(p-1/2)}{4\pi^{3/2}\Gamma(p)} \sum_{n\in\mathbb{Z}} |n-\Phi|^{1-2p} = \frac{\Gamma((1+\alpha)/2)}{4\pi^{3/2}\Gamma(1+\alpha/2)} \sum_{n\in\mathbb{Z}} |n-\Phi|^{-1-\alpha}.$$

Returning to the estimate in (21) we conclude that

$$\begin{aligned}\mathrm{Tr}(\widetilde{V}_+^{p/2}(-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2)^{-p}\widetilde{V}_+^{p/2}) &= g_{\Phi,p} \int_{\mathbb{R}} \int_{-\pi}^{\pi} \widetilde{V}(t,\theta)_+^p \,\mathrm{d}\theta\,\mathrm{d}t \\ &= g_{\Phi,p} \int_{\mathbb{R}^2} V(x)_+^p |x|^{2p-2}\,\mathrm{d}x\,,\end{aligned}$$

which completes the proof of (8) with the constant given in (10). This easily implies the upper bound in (9). The lower bound is a consequence of Remark 6. ☐

Remark 5. Despite employing a non-unitary transformation into logarithmic coordinates, the bound in (21) can be derived from the operator's form core image and an application of Glazman's Lemma. We refer to [12, Proposition 7.4] for a similar statement.

Remark 6. A standard argument shows that the sharp constants in the CLR-type inequality (8) and in the magnetic Hardy-Sobolev inequality (18) satisfy

$$S_{\Phi,q} \geqslant C_{\Phi,\alpha}^{-2/(\alpha+2)} \qquad \text{with } \frac{2}{\alpha+2} + \frac{2}{q} = 1\,, \tag{22}$$

see e.g. [12, Proposition 5.7]. In particular, (10) implies that

$$S_{\Phi,q} \geqslant \left(\frac{\Gamma(1/2 + 2/(q-2))}{4\pi^{3/2}\Gamma(1+2/(q-2))} \sum_{n\in\mathbb{Z}} |n-\Phi|^{-1-4/(q-2)} \right)^{-(q-2)/q}$$

and the upper bound in (9) implies that

$$S_{\Phi,q} \gtrsim_q d(\Phi)^{1+2/q}\,. \tag{23}$$

Let us show that this bound is optimal, that is,

$$S_{\Phi,q} \lesssim_q d(\Phi)^{1+2/q}\,. \tag{24}$$

In view of (22) this will prove the lower bound in (9) and thereby complete the proof of Theorem 1.

We fix $\varphi \in C_c^\infty(\mathbb{R})$ and define

$$u(r\cos\theta, r\sin\theta) = \varphi((\ln r)/\ell)\, e^{in\theta}\,,$$

where $n \in \mathbb{Z}$ is such that $d(\Phi) = |n-\Phi|$. Then, by (18) after changing to logarithmic coordinates,

$$\ell^{-1}\int_{\mathbb{R}} |\varphi'(t)|^2\,\mathrm{d}t + d(\Phi)^2\ell \int_{\mathbb{R}} |\varphi(t)|^2\,\mathrm{d}t \geqslant S_{\Phi,q}\left(\ell\int_{\mathbb{R}} |\varphi(t)|^q\,\mathrm{d}t\right)^{2/q}.$$

Choosing $\ell = d(\Phi)^{-1}$ we obtain (24).

2.2. **The antisymmetric operator.** The same construction and arguments carry over to the antisymmetric operator. In this case, the Hardy–Sobolev inequalities (18) are replaced by the inequalities

(25)
$$\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x \geqslant S_q \left(\int_{\mathbb{R}^2} \frac{|u|^q}{|x|^2} \, \mathrm{d}x \right)^{\frac{2}{q}} \qquad \text{for all antisymmetric } u \in C_c^\infty(\mathbb{R}^2 \backslash \{0\})$$

with $S_q > 0$ provided that $q \in [2, \infty)$. A proof of this inequality can be found in [15]. In the special case $q = 2$ we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x \geqslant \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, \mathrm{d}x \qquad \text{for all antisymmetric } u \in C_c^\infty(\mathbb{R}^2 \backslash \{0\}) \tag{26}$$

with the sharp constant equal to one.

For symmetric V such that either the right side in Theorem 3 or in Corollary 4 is finite we can define the operators $-\Delta_{\mathbf{as}} - V$ in $L^2_{\mathbf{as}}(\mathbb{R}^2)$ similarly as in the Aharonov–Bohm case.

Proof of Theorem 3. We fix $\alpha > 0$ and take $0 \leqslant V \in L^{1+\alpha/2}(\mathbb{R}^2; \mathrm{d}x/|x|^2)$ as before. The Birman–Schwinger operator $V^{1/2}(-\Delta_{\mathbf{as}})^{-1} V^{1/2}$ in $L^2_{\mathbf{as}}(\mathbb{R}^2)$ is unitarily equivalent to the operator $\widetilde{V}^{1/2}(-\partial_t^2 - \partial_\theta^2)^{-1} \widetilde{V}^{1/2}$ acting in the subspace of function $u \in L^2(\mathbb{R} \times \mathbb{S}^1)$ satisfying $u(t, \theta) = -u(t, \pi/2 - \theta)$. Here $\widetilde{V}$ is defined as in the proof of Theorem 3. Applying the Birman–Schwinger principle and the Lieb–Thirring inequality as before, we are reduced to finding the integral kernel $G_p(t, \theta; \tau, \vartheta)$ corresponding to $(-\partial_t^2 - \partial_\theta^2)^{-p}$ acting in this subspace. To find it, we argue as previously, using a Fourier decomposition in terms of the antisymmetric angular harmonics $\varphi_n(\theta) = \pi^{-1/2} \sin(n(\theta - \pi/4))$, $n \in \mathbb{N}$. It follows that

$$\begin{aligned} G_p(t, \theta; t, \theta) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \varphi_n(\theta)^2 \int_{\mathbb{R}} \frac{1}{(\xi^2 + n^2)^p} \, \mathrm{d}\xi \\ &\leqslant \frac{\Gamma(p - 1/2)}{2\pi^{3/2}\Gamma(p)} \left(\sum_{n=1}^{\infty} n^{1-2p} \right) = \frac{\Gamma(p - 1/2)}{2\pi^{3/2}\Gamma(p)} \zeta(2p - 1) \, , \end{aligned}$$

where ζ denotes the Riemann zeta function. This proves Theorem 3. ☐

3. Interpolation and proof of corollaries 2 and 4

In this section we derive Corollaries 2 and 4 from Theorems 1 and 3, respectively. We use a variant of an interpolation argument by Birman and Solomyak [5], but we avoid any explicit mention of interpolation theory or ideals of compact operators.

Proof of Corollary 2. We fix $\alpha > 0$ and recall that we may assume that $0 < \Phi \leqslant 1/2$ and that $V \geqslant 0$. With two parameters $s > 0$ and $0 < \theta < 1$ to be determined we write

$$D_\Phi^2 - V = \theta(D_\Phi^2 - \theta^{-1} s|x|^{-2}) + (1 - \theta)(D_\Phi^2 - (1 - \theta)^{-1}|x|^{-2}(|x|^2 V - s)) \, .$$

Assuming that $\theta^{-1} s \leqslant \Phi^2$ we can use the magnetic Hardy inequality (19) to bound

$$D_\Phi^2 - V \geqslant (1 - \theta)(D_\Phi^2 - (1 - \theta)^{-1}|x|^{-2}(|x|^2 V - s)_+) \, .$$

Thus, by the variational principle

$$N(D_\Phi^2 - V) \leqslant N(D_\Phi^2 - (1 - \theta)^{-1}|x|^{-2}(|x|^2 V - s)_+) \, .$$

For an arbitrary $0 < \beta < \alpha$ we can apply Theorem 1 and obtain

$$N(D_\Phi^2 - V) \leqslant C_{\Phi,\beta}(1-\theta)^{-1-\beta/2} \int_{\mathbb{R}^2} (|x|^2 V(x) - s)_+^{1+\beta/2} \frac{\mathrm{d}x}{|x|^2}\,.$$

Abbreviating $[V] := \sup_{t>0} t^{1+\alpha/2} \int_{|x|^2V(x)>t} \frac{\mathrm{d}x}{|x|^2}$ and using the layer cake representation we find

$$\begin{aligned}\int_{\mathbb{R}^2} (|x|^2V(x)-s)_+^{1+\beta/2} \frac{\mathrm{d}x}{|x|^2} &= (1+\beta/2) \int_0^\infty \int_{|x|^2V(x)-s>\sigma} \frac{\mathrm{d}x}{|x|^2}\, \sigma^{\beta/2}\, \mathrm{d}\sigma \\ &\leqslant (1+\beta/2)\,[V] \int_0^\infty (\sigma+s)^{-1-\alpha/2}\, \sigma^{\beta/2}\, \mathrm{d}\sigma \\ &= \frac{\Gamma(2+\beta/2)\,\Gamma((\alpha-\beta)/2)}{\Gamma(1+\alpha/2)}\, s^{(\beta-\alpha)/2}\,[V]\,.\end{aligned}$$

In the last computation we used a beta function identity. To minimize this bound, we choose $s = \theta\Phi^2$ and obtain

$$N(D_\Phi^2 - V) \leqslant \frac{\Phi^{\beta-\alpha}\, C_{\Phi,\beta}}{\sup_{0<\theta<1}(1-\theta)^{1+\beta/2}\theta^{(\alpha-\beta)/2}} \frac{\Gamma(2+\beta/2)\,\Gamma((\alpha-\beta)/2)}{\Gamma(1+\alpha/2)}\,[V]\,.$$

This bound can still be optimized with respect to $\beta \in (0,\alpha)$. This proves (11). Taking a fixed β (say $\beta = \alpha/2$) and recalling that $C_{\Phi,\beta} \lesssim_\beta \Phi^{-1-\beta}$ by (9), we deduce the upper bound in (12). The lower bound follows from (13) together with the lower bound in (9). $\square$

The proof of Corollary 4 is similar to that of Corollary 2 and is omitted.

4. Long-range potentials and behaviour of constants

In this section we construct, for arbitrary $\alpha > 0$, potentials V that saturate the weak bounds (11) and (17) in the strong coupling limit. We follow arguments that were carried out for dimensions $d \geqslant 3$ in [2,3,18].

Theorem 7. *Let $\Phi \in \mathbb{R}\backslash\mathbb{Z}$, let $p > 0$ and assume that $V \in L^\infty(\mathbb{R}^2)$ satisfies*

$$V(x) = |x|^{-2}(\ln|x|)^{-1/p}\,(1+o(1)) \qquad \text{as } |x| \to \infty\,.$$

Then for $p > 1$

$$\lim_{\lambda\to\infty} \lambda^{-p} N(D_\Phi^2 - \lambda V) = \frac{\Gamma(p-1/2)}{2\sqrt{\pi}\Gamma(p)} \sum_{n\in\mathbb{Z}} \frac{1}{|n-\Phi|^{2p-1}}\,,$$

for $p = 1$

$$\lim_{\lambda\to\infty} (\lambda\ln\lambda)^{-1} N(D_\Phi^2 - \lambda V) = \frac{1}{2}\,,$$

and for $p < 1$

$$\lim_{\lambda\to\infty} \lambda^{-1} N(D_\Phi^2 - \lambda V) = \frac{1}{4\pi} \int_{\mathbb{R}^2} V(x)_+\, \mathrm{d}x\,.$$

In the theorem we clearly see the difference between the long range case $p \geqslant 1$ and the short range case $p < 1$. In the former case the asymptotics are insensitive to the local behavior of V and solely determined by its asymptotic behavior, while in the latter case they are essentially determined by the local behavior of V.

We note that if V is as in the theorem with $p > 1$, then with $\alpha = 2(p-1)$

$$\lim_{\lambda\to\infty} \lambda^{-p} \sup_{t>0} t^{1+\alpha/2} \int_{\lambda|x|^2V(x)_+>t} \frac{\mathrm{d}x}{|x|^2} = \sup_{t>0} t^p \int_{|x|^2V(x)_+>t} \frac{\mathrm{d}x}{|x|^2} \in (0,\infty)\,.$$

Therefore Theorem 7 shows that the weak bounds (11) are saturated for the potentials λV as $\lambda \to \infty$.

Moreover, the asymptotics for $p = 1$ show that one cannot expect to have a version of the weak inequality (11) that is homogeneous of degree one in V.

Remark 8. For comparison, if $\Phi = 0$ and V is as in Theorem 7 with $p > 1/2$ then $N(-\Delta - \lambda V) = \infty$ for all $\lambda > 0$. The same holds for $p = 1/2$ provided $\lambda > 1/4$; see [12, Proposition 4.21].

Proof. We mostly focus on the case $p \geqslant 1$ and discuss the case $p < 1$ at the end. Let W_p be defined as

$$W_p(x) := \begin{cases} |x|^{-2}(\ln|x|)^{-1/p}, & |x| > e\,, \\ 0, & |x| \leqslant e\,. \end{cases} \tag{27}$$

We will prove the theorem for $p \geqslant 1$ in the special case $V = W_p$. By simple approximation arguments, this implies the result in the general case.

We start by simplifying the problem. Consider the restriction of the operator $D_\Phi^2 - \lambda W_p$ to the region $\{x\colon |x| > e\}$ with Dirichlet and Neumann boundary conditions, denoted by $H_\Phi^D(\lambda W_p)$ and $H_\Phi^N(\lambda W_p)$, respectively. Then, since $W_p \equiv 0$ for $|x| \leqslant e$, by the variational principle,

$$N(H_\Phi^D(\lambda W_p)) \leqslant N(D_\Phi^2 - \lambda W_p) \leqslant N(H_\Phi^N(\lambda W_p))\,. \tag{28}$$

It follows, using logarithmic-coordinates $r = e^{t+1}$ and the definition of W_p, that we need only estimate the number of negative eigenvalues of the operator

$$-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2 - \lambda(t+1)^{-1/p} \text{ in } L^2((0,\infty)\times\mathbb{S}^1),$$

from above and below, where the operator is considered with Neumann and Dirichlet boundary conditions at $t = 0$, respectively.

Now we carry out a further bracketing argument. We fix $L > 0$ and for $k \in \mathbb{N}_0$ denote by $H_{k,L}^D(V)$ and $H_{k,L}^N(V)$ the restrictions of $-\partial_t^2 + (\mathrm{i}\partial_\theta - \Phi)^2 - V(t)$ to the intervals $(kL,(k+1)L)$ with Dirichlet and Neumann boundary conditions respectively. Then, using $((k+1)L+1)^{-1/p} \leqslant (t+1)^{-1/p} \leqslant (kL+1)^{-1/p}$ on $(kL,(k+1)L)$,

(29)

$$N(H_\Phi^D(\lambda W_p)) \geqslant \sum_{k=0}^\infty N(H_{k,L}^D(\lambda(t+1)^{-1/p})) \geqslant \sum_{k=0}^\infty N(H_{k,L}^D(\lambda((k+1)L+1)^{-1/p}))$$

and

$$N(H_\Phi^N(\lambda W_p)) \leqslant \sum_{k=0}^\infty N(H_{k,L}^N(\lambda(t+1)^{-1/p})) \leqslant \sum_{k=0}^\infty N(H_{k,L}^N(\lambda(kL+1)^{-1/p}))\,. \tag{30}$$

It remains to estimate each of these, where we first consider the case of $p > 1$. Starting with the lower bound, we use (29) to see that

$$\begin{aligned} N(H_\Phi^D(\lambda W_p)) &\geqslant \sum_{k=0}^\infty \#\{(m,n)\in\mathbb{N}\times\mathbb{Z}\colon \frac{\pi^2 m^2}{L^2} + (n-\Phi)^2 < \lambda((k+1)L+1)^{-1/p})\} \\ &\geqslant \sum_{m\in\mathbb{N},n\in\mathbb{Z}} \left(L^{-1}\lambda^p\left(\pi^2m^2/L^2 + (n-\Phi)^2\right)^{-p} - 1 - L^{-1}\right)_+ = (\mathrm{I}) + (\mathrm{II}), \end{aligned}$$

where

$$
\begin{aligned}
(\mathrm{I}) &= \sum_{m\in\mathbb{N}_0, n\in\mathbb{Z}} \left(L^{-1}\lambda^p\left(\pi^2 m^2/L^2 + (n-\Phi)^2\right)^{-p} - 1 - L^{-1}\right)_+ \\
&\geqslant \lambda^p \sum_{n\in\mathbb{Z}} \int_0^\infty \left(\left(\pi^2\tau^2 + (n-\Phi)^2\right)^{-p} - \lambda^{-p}(L+1)\right)_+ \mathrm{d}\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{II}) &= -\sum_{n\in\mathbb{Z}} \left(L^{-1}\lambda^p|n-\Phi|^{-2p} - 1 - L^{-1}\right)_+ \\
&\geqslant -L^{-1}\lambda^p \sum_{n\in\mathbb{Z}} |n-\Phi|^{-2p}.
\end{aligned}
$$

Meanwhile, for the upper-bound (30) we find that

$$
\begin{aligned}
N(H_\Phi^N(\lambda W_p)) &\leqslant \sum_{k=0}^\infty \#\{(m,n)\in\mathbb{N}_0\times\mathbb{Z}\colon \frac{\pi^2 m^2}{L^2} + (n-\Phi)^2 < \lambda(kL+1)^{-1/p}\} \\
&= (\mathrm{III}) + (\mathrm{IV}),
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathrm{III}) &= \#\{(m,n)\in\mathbb{N}_0\times\mathbb{Z}\colon \frac{\pi^2 m^2}{L^2} + (n-\Phi)^2 < \lambda\} \\
&\leqslant \#\{n\in\mathbb{Z}\colon (n-\Phi)^2 < \lambda\} + \sum_{n\in\mathbb{Z}} \pi^{-1}L\left(\lambda-(n-\Phi)^2\right)_+^{1/2} \\
&\leqslant (2\sqrt{\lambda}+1) + 2\pi^{-1}L(\lambda-\Phi^2)_+^{1/2} + \pi^{-1}L\int_{\mathbb{R}} (\lambda-(t-\Phi)^2)_+^{1/2}\,\mathrm{d}t \\
&= (2\sqrt{\lambda}+1) + 2\pi^{-1}L(\lambda-\Phi^2)_+^{1/2} + 2^{-1}L\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{IV}) &= \sum_{k=1}^\infty \#\{(m,n)\in\mathbb{N}_0\times\mathbb{Z}\colon \frac{\pi^2 m^2}{L^2} + (n-\Phi)^2 < \lambda(kL+1)^{-1/p}\} \\
&\leqslant \sum_{m\in\mathbb{N}_0, n\in\mathbb{Z}} \left(L^{-1}\lambda^p\left(\pi^2 m^2/L^2 + (n-\Phi)^2\right)^{-p} - L^{-1}\right)_+ \\
&\leqslant \lambda^p \sum_{n\in\mathbb{Z}} \int_0^\infty \left(\left(\pi^2\tau^2 + (n-\Phi)^2\right)^{-p} - \lambda^{-p}\right)_+ \mathrm{d}\tau.
\end{aligned}
$$

Taking the limsup and liminf as $\lambda\to\infty$ and then the limit $L\to\infty$, we find

$$
\begin{aligned}
\liminf_{\lambda\to\infty} \lambda^{-p} N(H_\Phi^D(\lambda W_p)) &\geqslant \sum_{n\in\mathbb{Z}} \int_0^\infty \left(\pi^2\tau^2 + (n-\Phi)^2\right)^{-p} \mathrm{d}\tau \\
&= \frac{\Gamma(p-1/2)}{2\sqrt{\pi}\Gamma(p)} \sum_{n\in\mathbb{Z}} \frac{1}{|n-\Phi|^{2p-1}},
\end{aligned}
$$

and similarly

$$
\limsup_{\lambda\to\infty} \lambda^{-p} N(H_\Phi^N(\lambda W_p)) \leqslant \frac{\Gamma(p-1/2)}{2\sqrt{\pi}\Gamma(p)} \sum_{n\in\mathbb{Z}} \frac{1}{|n-\Phi|^{2p-1}}.
$$

This proves the claimed limit for $p>1$.

For the case of $p = 1$, we carefully consider the terms that produce a logarithmic divergence. In this case, the choice of intervals does not matter, so we take $L = 1$. We start by using (29) to find that

$$\begin{aligned} N(H_\Phi^D(\lambda W_1)) &\geqslant \lambda \sum_{m\in\mathbb{N},n\in\mathbb{Z}} \left(\left(\pi^2 m^2 + (n-\Phi)^2\right)^{-1} - 2\lambda^{-1}\right)_+ \\ &\geqslant \lambda \int_{\mathbb{R}\setminus(-1,1)} \int_1^\infty \left(\left(\pi^2\tau^2 + (t-\Phi)^2\right)^{-1} - 2\lambda^{-1}\right)_+ \mathrm{d}\tau\, \mathrm{d}t - O(\lambda) \\ &\geqslant \lambda(2\pi)^{-1} \iint\limits_{\sigma^2+s^2>R_1^2} \left(\left(\sigma^2+s^2\right)^{-1} - 2\lambda^{-1}\right)_+ \mathrm{d}\sigma\, \mathrm{d}s - O(\lambda), \end{aligned}$$

with $R_1^2 := (\pi^2 + (1-\Phi)^2)/2$. When passing to the last line we increased the region of integration in the first term, noting that additional integral is $O(\lambda)$. For the upper bound (30), in the decomposition above, the term (III) is of order $O(\lambda)$ as $\lambda \to \infty$, thus we see that

$$\begin{aligned} N(H_\Phi^N(\lambda W_1)) &\leqslant O(\lambda) + \text{(IV)} \\ &= \sum_{m\in\mathbb{N}_0,n\in\mathbb{Z}} \#\{k\in\mathbb{N}\colon k < \lambda\left(\pi^2m^2 + (n-\Phi)^2\right)^{-1} - 1\} + O(\lambda) \\ &= \lambda \sum_{m\in\mathbb{N}\setminus\{1\},n\in\mathbb{Z}} \left(\left(\pi^2m^2 + (n-\Phi)^2\right)^{-1} - \lambda^{-1}\right)_+ + O(\lambda) \\ &\leqslant \lambda(2\pi)^{-1} \iint\limits_{\sigma^2+s^2>R_2^2} ((\sigma^2+s^2)^{-1} - \lambda^{-1})_+ \,\mathrm{d}\sigma\, \mathrm{d}s + O(\lambda) \end{aligned}$$

with $R_2^2 := \pi^2 + (1-\Phi)^2$. For $R = R_1, R_2$ we compute

$$\iint\limits_{\sigma^2+s^2>R^2} ((\sigma^2+s^2)^{-1} - \lambda^{-1})_+ \,\mathrm{d}\sigma\, \mathrm{d}s = 2\pi \int_R^{\sqrt{\lambda}} (r^{-2} - \lambda^{-1}) r\, \mathrm{d}r = \pi \ln \lambda + O(1)\,.$$

This proves the claimed limit for $p = 1$.

Finally, we comment on the case $p < 1$. We clearly have

$$\liminf_{\lambda\to\infty} \lambda^{-1} N(D_\Phi^2 - \lambda V) \geqslant \frac{1}{4\pi}\int_{\mathbb{R}^2} V(x)_+\, dx\,. \tag{31}$$

Indeed, for given, sufficiently large $R > 0$ we bound $V \geqslant V\mathbb{1}(|x| < R)$ (here we use that V is nonnegative outside of a bounded set) and then impose a Dirichlet condition at $|x| = R$ to bound $N(D_\Phi^2 - \lambda V)$ from below by the number of negative eigenvalues of the corresponding Dirichlet operator on $\{|x| < R\}$. By [11, Corollary 1.2], and the works [9, 28], for the latter operator one has Weyl asymptotics. Since $R > 0$ can be chosen arbitrarily large, we obtain (31). Let us explain in some more detail how to obtain the Weyl asymptotics from references [11] and [9]. In view of [9, Theorem 2.2] the bound in [11, Theorem 1.1] remains valid in the presence of a magnetic field, at the expense of an increase of the constant by a universal factor. With this bound at hand, one can follow the proof of [11, Corollary 1.2] and obtain the asymptotics in the presence of a magnetic field; see also [10].

To prove

$$\limsup_{\lambda\to\infty} \lambda^{-1} N(D_\Phi^2 - \lambda V) \leqslant \frac{1}{4\pi}\int_{\mathbb{R}^2} V(x)_+\, dx\,, \tag{32}$$

we set, for all $R>1$, $\widetilde{W}_p(x)=\mathbb{1}(|x|>R)|x|^{-2}(\ln|x|)^{-1/p}+\mathbb{1}(|x|\leqslant R)R^{-2}(\ln R)^{-1/p}$. For $0<\theta<1$ and $\varepsilon>0$ we decompose

$$D_\Phi^2-\lambda V=\theta\left(D_\Phi^2-\theta^{-1}(1+\varepsilon)\lambda\widetilde{W}_p\right)+(1-\theta)\left(D_\Phi^2-(1-\theta)^{-1}\lambda(V-(1+\varepsilon)\widetilde{W}_p)\right)$$

and obtain

$$N(D_\Phi^2-\lambda V)\leqslant N(D_\Phi^2-\theta^{-1}(1+\varepsilon)\lambda\widetilde{W}_p)+N(D_\Phi^2-(1-\theta)^{-1}\lambda(V-(1+\varepsilon)\widetilde{W}_p)_+)\,.$$

Since $\widetilde{W}_p$ is radially nonincreasing, it results from either (6) or (7) that

$$N(D_\Phi^2-\theta^{-1}(1+\varepsilon)\lambda\widetilde{W}_p)\lesssim_\Phi\theta^{-1}(1+\varepsilon)\lambda\int_{\mathbb{R}^2}\widetilde{W}_p\,\mathrm{d}x\lesssim_{\Phi,p}\theta^{-1}(1+\varepsilon)(\ln R)^{1-1/p}\lambda\,.$$

Meanwhile, by assumption there is an $R_\varepsilon<\infty$ such that for all $|x|\geqslant R_\varepsilon$ one has $V(x)\leqslant(1+\varepsilon)|x|^{-2}(\ln|x|)^{-1/p}$. Therefore, the potential $(V-(1+\varepsilon)\widetilde{W}_p)_+$ is supported in a ball and with the help of [30] one finds

$$\lim_{\lambda\to\infty}\lambda^{-1}N(D_\Phi^2-(1-\theta)^{-1}\lambda(V-(1+\varepsilon)\widetilde{W}_p)_+)=\frac{1}{4\pi}(1-\theta)^{-1}\int_{\mathbb{R}^2}(V-(1+\varepsilon)\widetilde{W}_p)_+\,\mathrm{d}x\,.$$

Thus, we have shown that

$$\limsup_{\lambda\to\infty}\lambda^{-1}N(D_\Phi^2-\lambda V)\leqslant\frac{1}{4\pi}(1-\theta)^{-1}\int_{\mathbb{R}^2}(V-(1+\varepsilon)\widetilde{W}_p)_+\,\mathrm{d}x+C_{\Phi,p}\theta^{-1}(1+\varepsilon)(\ln R)^{1-1/p}.$$

Letting $R\to\infty$ using the integrability of V and $p<1$, we obtain

$$\limsup_{\lambda\to\infty}\lambda^{-1}N(D_\Phi^2-\lambda V)\leqslant\frac{1}{4\pi}(1-\theta)^{-1}\int_{\mathbb{R}^2}V_+\,\mathrm{d}x\,.$$

Since $\theta\in(0,1)$ is arbitrary, we obtain (32). This concludes the proof. □

Remark 9. Let us use Theorem 7 to prove the lower bound (12) on $C'_{\Phi,\alpha}$. Let $\alpha>0$ and $p=1+\alpha/2>1$, then for W_p as in the proof of Theorem 7

$$\lim_{\lambda\to\infty}\lambda^{-p}\sup_{t>0}t^p\int_{\lambda W_p|x|^2>t}\frac{\mathrm{d}x}{|x|^2}=\sup_{t>0}t^p\int_{W_p|x|^2>t}\frac{\mathrm{d}x}{|x|^2}=2\pi,$$

and thus, by the asymptotic formula in Theorem 7,

$$C'_{\Phi,\alpha}\geqslant\lim_{\lambda\to\infty}\frac{N(D_\Phi^2-\lambda W_p)}{\sup_{t>0}t^p\int_{\lambda W_p|x|^2>t}\frac{\mathrm{d}x}{|x|^2}}=\frac{\Gamma(\alpha/2+1/2)}{4\pi^{3/2}\Gamma(1+\alpha/2)}\sum_{n\in\mathbb{Z}}|n-\Phi|^{-1-\alpha}.$$

This proves (12).

Finally, we note that the corresponding results hold in the antisymmetric case by near identical argument. We state them below without proof.

Theorem 10. *Let $p\geqslant1$ and let V be as in Theorem 7. Then for $p>1$*

$$\lim_{\lambda\to\infty}\lambda^{-p}N(-\Delta_{\mathbf{as}}-\lambda V)=\frac{\Gamma(p-1/2)}{2\sqrt{\pi}\Gamma(p)}\zeta(2p-1)$$

and for $p=1$

$$\lim_{\lambda\to\infty}(\lambda\ln\lambda)^{-1}N(-\Delta_{\mathbf{as}}-\lambda V)=\frac{1}{4}\,.$$

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