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# Dirichlet and Neumann Eigenvalue Problems on Domains in Euclidean Spaces*

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We obtain here some inequalities for the eigenvalues of Dirichlet and Neumann value problems for general classes of operators (or system of operators) acting in $L^2(\Omega)$ (or $L^2(\Omega, \mathbb{C}^m)$), $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$. 

## 1. INTRODUCTION

Let $\Omega$ be an open domain in $\mathbb{R}^d$, $d \geqslant 1$, and $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ be the eigenvalues of the Dirichlet boundary problem for the Laplace operator $-\Delta^{\mathscr{D}}$ in $\Omega$. Denote by $|\Omega|$ the Lebesgue measure of the domain $\Omega$ and by $L_d^{cl} = v_d (2\pi)^{-d} = 2^{-d} \pi^{-d/2} / \Gamma(1 + d/2)$, where $v_d$ is the volume of the unit ball in $\mathbb{R}^d$. Li and Yau [LY] proved that the eigenvalues $\lambda_k$ satisfy the inequality

$$\lambda_k \geqslant \frac{d}{d+2} (L_d^{cl} \, |\Omega|)^{-2/d} k^{2/d}, \qquad \forall k \in \mathbb{N}. \tag{1.1}$$

The constant $L_d^{cl}$, the so called "classical constant," appears in the Weyl asymptotic formula for the counting function of eigenvalues. The proof of (1.1) is based on a sharp inequality concerning the sum of the first eigenvalues

$$\sum_{j=1}^{k} \lambda_j \geqslant \frac{d}{d+2} (L_d^{cl} \, |\Omega|)^{-2/d} k^{1+2/d}, \qquad \forall k \in \mathbb{N}. \tag{1.2}$$

The constant on the right hand side of (1.2) cannot be improved because it coincides with the asymptotical constant for the sum in the left hand side of (1.2) as $k \to \infty$.

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An opposite inequality can be obtained for the eigenvalues of the Neumann boundary problem. Let $0 = \mu_1 < \mu_2 \leqslant \cdots$ be the eigenvalues of the Neumann Laplacian $-\Delta^{\mathcal{N}}$ in a bounded domain $\Omega$ with piecewise smooth boundary. By adapting the approach of Li and Yau to this problem, Kröger [K1] proved the upper estimate

$$\mu_{k+1} \leqslant \left(\frac{d+2}{d}\right)^{2/d} (L_d^{cl}\,|\Omega|)^{-2/d}\, k^{2/d}, \qquad \forall k \in \mathbb{N}. \tag{1.3}$$

The key inequality here was the upper estimate for the sum of the first eigenvalues $\mu_j$'s

$$\sum_{k=1}^{k} \mu_k \leqslant \frac{d}{d+2} (L_d^{cl}\,|\Omega|)^{-2/d}\, k^{1+2/d}, \qquad \forall k \in \mathbb{N}.$$

In this paper we show that the inequalities (1.1) and (1.3) are corollaries of general (sharp) trace inequalities for convex functions of operators. In particular, (1.1) and (1.3) can be extended to the Dirichlet and Neumann boundary problems for various classes of (systems of) differential and pseudo-differential operators with constant coefficients (for example $(-\Delta)^{\alpha}$, $\alpha > 0$, operator of classical elasticity, etc). This approach can be also easily extended to operators acting on functions with values in a Hilbert space. We shall not consider this case here only because it requires many additional notations and assumptions.

Notice that the inequality $\lambda_k \geqslant C_d\,|\Omega|^{-2/d}\, k^{2/d}$ with a constant $C_d < d/(d+2)(L_d^{cl})^{-2/d}$ was proved for bounded domains in [BS, C] and later for arbitrary domains in [R1, 2, M, Lb1] (see also [L]).

G. Pólya conjectured in [P] that (1.1) should hold without the multiplier $d/(d+2)$. He proved this conjecture for "tiling" domains $\Omega \subset \mathbb{R}^2$, i.e., copies of $\Omega$ fill the plane without gaps. In Subsection 2.3 we notice that Theorem 2.1 allows us to justify this conjecture for domains $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1 + d_2 = d$, $d_1 \geqslant 2$, $d_2 \geqslant 1$, as long as the Dirichlet Laplacian in $L^2(\Omega_1)$ satisfies the Pólya conjecture and $\Omega_2$ is an arbitrary domain whose $d_2$-Lebesgue measure is finite (see Theorem 2.8 and Corollary 2.9).

In [LP] the method of [LY] was applied to the Dirichlet boundary problem for (systems of) differential operators of a higher order. The method presented here, however, allows us to obtain the same constants for differential operators and better constants than in [LP] for systems of differential operators (see Corollary 2.9 and Remark 2.10).

In Section 4 we obtain some more inequalities on the eigenvalues of $-\Delta^{\mathcal{D}}$ and, in particular, we give an upper bound for the eigenvalues $\lambda_k$, assuming only that the spectrum of $-\Delta^{\mathcal{D}}$ in $L_2(\Omega)$ is discrete.

In what follows we shall be dealing with different classes of vector functions on $\mathbb{R}^d$ with values in $\mathbb{C}^m$, $\mathbb{R}_+ = (0, +\infty)$, $D = -i\partial/\partial x$. By $\varphi_\lambda$ we denote the convex function

$$\varphi_\lambda(t) = (\lambda - t)_+ = \begin{cases} \lambda - t, & t < \lambda, \\ 0, & t \geqslant \lambda. \end{cases}$$

Assuming that a selfadjoint operator $B \geqslant 0$ has a discrete spectrum accumulating at infinity, we denote by $N(\lambda, B)$ its counting function of the spectrum

$$N(\lambda, B) = \#\{k : \lambda_k < \lambda\}.$$

If $A$ is an $m \times m$ complex matrix, then $A^*$ is its adjoint matrix.

## 2. DIRICHLET BOUNDARY VALUE PROBLEM

**1.** Let $\Omega$ be an open subset of $\mathbb{R}^d$ of finite measure. We shall deal with various classes of functions with values in $\mathbb{C}^m$, $m \in \mathbb{N}$. The norm and the scalar product in $\mathbb{C}^m$ is denoted by $\|\cdot\|$ and $(\cdot, \cdot)$ respectively. Let

$$L^2(\Omega, \mathbb{C}^m) = \left\{ u : \int_\Omega \|u(x)\|^2 \, dx < \infty \right\}.$$

The class of smooth vector valued functions with compact support, $C_0^\infty(\Omega, \mathbb{C}^m) \subset L^2(\Omega, \mathbb{C}^m)$, is dense in $L^2(\Omega, \mathbb{C}^m)$.

Let $A(\xi)$ be a complex $m \times m$ measurable matrix function, $\xi \in \mathbb{R}^d$. We assume for simplicity, that there is $\kappa \in \mathbb{R}_+$ and a constant $C$ such that

$$0 \leqslant \|A(\xi)\| \leqslant C\, |\xi|^\kappa, \qquad \xi \in \mathbb{R}^d. \tag{2.1}$$

Let $\hat{u}$ be the Fourier transform of the vector function $u \in L^2(\mathbb{R}^d, \mathbb{C}^m)$. We introduce a sesquilinear from $\mathfrak{B}_\Omega$ defined on the vector functions from the class $C_0^\infty(\Omega, \mathbb{C}^m)$,

$$\mathfrak{B}_\Omega[u, v] = (2\pi)^{-d} \int_{\mathbb{R}^d} (A(\xi)\, \hat{u}(\xi), A(\xi)\, \hat{v}(\xi))\, d\xi$$

$$= (2\pi)^{-d} \int_{\mathbb{R}^d} (B(\xi)\, \hat{u}(\xi), \hat{v}(\xi))\, d\xi, \qquad u, v \in C_0^\infty(\Omega, \mathbb{C}^m), \tag{2.2}$$

where $B(\xi) = A^*(\xi)\, A(\xi)$. The completion of the class $C_0^\infty(\Omega, \mathbb{C}^m)$ with respect to the quadratic form $\mathfrak{B}_\Omega[u, u] + \int_\Omega \|u\|^2 \, dx$, defines a Hilbert space $\mathscr{H}[\mathfrak{B}_\Omega] \subset L^2(\Omega, \mathbb{C}^m)$. From (2.1) it follows that the Sobolev space $H_0^\kappa(\Omega, \mathbb{C}^m)$ is a subspace of $\mathscr{H}[\mathfrak{B}_\Omega]$. The closed quadratic form $\mathfrak{B}_\Omega$

defined on $\mathscr{H}[\mathfrak{B}_\Omega]$ gives a selfadjoint "pseudodifferential operator" with constant coefficients which we denote by $B_{\mathscr{D}}$.

If $\Omega = \mathbb{R}^d$, then the above construction leads to a closed quadratic form $\mathfrak{B}_{\mathbb{R}^d}$. The selfadjoint operator defined by $\mathfrak{B}_{\mathbb{R}^d}$ is denoted by $B$. Both operators $B_{\mathscr{D}}$ and $B$ can be considered as the Friedrichs extension of the pseudodifferential operator

$$B_0(D)\, u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\,\xi} B(\xi)\, u(y)\, dy\, d\xi$$

defined on the intersection $C_0^\infty(\mathbb{R}^d, \mathbb{C}^m) \cap L^2(\Omega, \mathbb{C}^m)$ and $C_0^\infty(\mathbb{R}^d, \mathbb{C}^m)$, respectively. We naturally identify the extension $B_{\mathscr{D}}$ with Dirichlet boundary value problem for $B$ in $\Omega$.

The next statement deals with a trace type inequality which is a special case of the Berezin–Lieb inequality (see [Bz1,2, Lb2, S] and for its generalizations [LS]). We include the proof of this statement for the sake of completeness.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$ and let the spectrum of the operator $B_{\mathscr{D}}$ consist of eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ such that $\lambda_k \to \infty$ as $k \to \infty$. Then the following inequality holds*:

$$\begin{aligned} \operatorname{Tr} \varphi_\lambda(B_{\mathscr{D}}) &= \sum_k (\lambda - \lambda_k)_+ \\ &\leqslant (2\pi)^{-d}\, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr}(\lambda - (B(\xi)))_+ \, d\xi \\ &= (2\pi)^{-d}\, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\lambda(B(\xi))\, d\xi, \qquad \lambda > 0. \end{aligned} \tag{2.3}$$

*Proof.* Let $\omega_1, \omega_2, ...$ be the orthonormal basis in $L^2(\Omega, \mathbb{C}^m)$ consisting of the eigenfunctions of the operator $B_{\mathscr{D}}$ whose corresponding eigenvalues are $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ and let $I$ be the unit matrix in $\mathbb{C}^m$. Then

$$\begin{aligned} \sum_k (\lambda - \lambda_k)_+ &= \sum_k (\lambda - (\mathfrak{B}_\Omega[\omega_k, \omega_k]))_+ \\ &= \sum_k \left( (2\pi)^{-d} \int_{\mathbb{R}^d} ((\lambda I - B(\xi))\, \hat{\omega}_k(\xi), \hat{\omega}_k(\xi))\, d\xi \right)_+ \\ &\leqslant (2\pi)^{-d} \sum_k \int_{\mathbb{R}^d} ((\lambda I - B(\xi))\, \hat{\omega}_k(\xi), \hat{\omega}_k(\xi))_+ \, d\xi. \end{aligned} \tag{2.4}$$

Denote by $\{\nu_j(\xi)\}_{j=1}^m$ and $\{\tau_j(\xi)\}_{j=1}^m$ the eigenvalues and the eigenvectors of the matrix $B(\xi)$ which are chosen to be measurable. The right hand side of (2.4) is less or equal than

$$\begin{aligned}
&(2\pi)^{-d} \sum_{j=1}^{m} \sum_{k} \int_{\mathbb{R}^d} (\lambda - \nu_j(\xi))_+ \, |(\hat{\omega}_k(\xi), \tau_j(\xi))|^2 \, d\xi \\
&\quad = (2\pi)^{-d} \sum_{j=1}^{m} \sum_{k} \int_{\mathbb{R}^d} (\lambda - \nu_j(\xi))_+ \left| \int_{\Omega} (e^{i(x,\,\xi)} \tau_j(\xi), \omega_k(x)) \, dx \right|^2 d\xi \\
&\quad = (2\pi)^{-d} \sum_{j=1}^{m} \int_{\mathbb{R}^d} (\lambda - \nu_j(\xi))_+ \, \| e^{i(\cdot,\,\xi)} \tau_j(\xi) \|^2_{L^2(\Omega,\, \mathbb{C}^m)} \, d\xi \\
&\quad = (2\pi)^{-d} \, |\Omega| \sum_{j=1}^{m} \int_{\mathbb{R}^d} (\lambda - \nu_j(\xi))_+ \, d\xi \\
&\quad = (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr}((\lambda I - B(\xi))_+) \, d\xi.
\end{aligned}$$

The proof is complete. ∎

*Remark.* The proof of Theorem 2.1 remains almost the same if instead of $\mathbb{C}^m$ we consider an infinite dimensional Hilbert space $H$.

DEFINITION 2.2. We say that $B(\xi)$ is a *positively homogeneous symbol of degree* $\alpha$, $\alpha > 0$, if there exists a family of unitary in $\mathbb{C}^m$ matrix-functions, $U(\lambda, \xi)$, such that

$$B(\lambda\xi) = \lambda^{\alpha} U^*(\lambda, \xi) \, B(\xi) \, U(\lambda, \xi), \qquad \lambda > 0.$$

If $B(\xi)$ is now a homogeneous symbol, then

$$\begin{aligned}
\operatorname{Tr} \varphi_\lambda(B(\xi)) &= \lambda \operatorname{Tr} \varphi_1(\lambda^{-1} B(\xi)) \\
&= \lambda \operatorname{Tr} \varphi_1(U(\lambda^{-1/\alpha}, \xi) \, B(\lambda^{-1/\alpha}\xi) \, U^*(\lambda^{-1/\alpha}, \xi)) \\
&= \lambda \operatorname{Tr} \varphi_1(B(\lambda^{-1/\alpha}\xi)).
\end{aligned}$$

If we integrate both sides of the last equality with respect to $\xi$ and change the variables $\lambda^{-1/\alpha}\xi \to \xi$, then we derive the following statement:

COROLLARY 2.3. *Let $B(\xi)$ be a positively homogeneous symbol of degree $\alpha$. Then under the conditions of Theorem* 2.1 *we obtain*

$$\sum_{k} (\lambda - \lambda_k)_+ \leqslant \lambda^{1 + d/\alpha} (2\pi)^{-d} \, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi)) \, d\xi, \qquad \lambda > 0. \quad (2.5)$$

*Remark* 2.4. The constant on the right hand side of (2.5) is the best possible since it appears in the corresponding asymptotic formula for $\sum_k (\lambda-\lambda_k)_+$, as $\lambda\to\infty$.

**2.** We use the results of Subsection 2.1 in order to deduce an upper estimate for the counting function of the spectrum of the operator $B_{\mathscr{D}}$.

THEOREM 2.5. *Let $\Omega\subset\mathbb{R}^d$ be an open set of finite measure, let $|\Omega|<\infty$, and let $B(\xi)$ be a positively homogeneous symbol of degree $\alpha$. Then*

$$N(\lambda, B_{\mathscr{D}}) \leqslant \lambda^{d/\alpha}(2\pi)^{-d}\,|\Omega|\,\frac{d}{\alpha}\left(1+\frac{\alpha}{d}\right)^{1+d/\alpha}\int_{\mathbb{R}^d}\operatorname{Tr}\varphi_1(B(\xi))\,d\xi. \tag{2.6}$$

*Proof.* Obviously

$$\begin{aligned} N(\eta-\rho, B_{\mathscr{D}}) &\leqslant \frac{1}{\rho}\int_0^\infty (\eta-\nu)_+\,dN(\nu, B_{\mathscr{D}})\\ &=\frac{1}{\rho}\sum_k (\eta-\lambda_k)_+, \qquad \eta\geqslant\rho>0. \end{aligned}$$

Therefore Corollary 2.3 implies that

$$N(\eta-\rho, B_{\mathscr{D}}) \leqslant \frac{\eta^{1+d/\alpha}}{\rho}(2\pi)^{-d}\,|\Omega|\int_{\mathbb{R}^d}\operatorname{Tr}\varphi_1(B(\xi))\,d\xi.$$

Choose $\eta=(1+\tau)\,\lambda$ and $\rho=\tau\lambda$. Then

$$N(\lambda, B_{\mathscr{D}}) \leqslant \lambda^{d/\alpha}\frac{(1+\tau)^{1+d/\alpha}}{\tau}(2\pi)^{-d}\,|\Omega|\int_{\mathbb{R}^d}\operatorname{Tr}\varphi_1(B(\xi))\,d\xi. \tag{2.7}$$

The minimum value of $(1+\tau)^{1+d/\alpha}\,\tau^{-1}$ is reached at $\tau=\alpha/d$. By substituting this value in (2.7) we obtain (2.6). ∎

Let $m=1$ and $B(\xi)=|\xi|^\alpha$. Then the operator $B_{\mathscr{D}}$ coincides with the operator of Dirichlet boundary problem for $(-\Delta)^{\alpha/2}$. In this case

$$(2\pi)^{-d}\frac{d}{\alpha}\left(1+\frac{\alpha}{d}\right)^{1+d/\alpha}\int_{\mathbb{R}^d}\operatorname{Tr}\varphi_1(B(\xi))\,d\xi = L_d^{cl}\left(1+\frac{\alpha}{d}\right)^{d/\alpha}$$

and we obtain

COROLLARY 2.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$. Then*

$$N(\lambda, ((-\Delta)^{\alpha/2})_{\mathscr{D}}) \leqslant \lambda^{d/\alpha} L_d^{cl} \left(1 + \frac{\alpha}{d}\right)^{d/\alpha} |\Omega|. \tag{2.8}$$

*Remark* 2.7. If $\alpha = 2$, then (2.8) is equivalent to the inequality (1.1) proved by Li and Yau in [LY].

**3.** We show here that in some special cases Theorem 2.1 implies the Pólya conjecture.

THEOREM 2.8. *Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, where $d_1 + d_2 = d$, $d_1 \geqslant 2$, $d_2 \geqslant 1$. Suppose that the operator of the Dirichlet boundary problem in $L^2(\Omega_1)$ satisfies the Pólya conjecture and $\Omega_2$ is an arbitrary domain whose $d_2$-Lebesgue measure is finite. Then*

$$N(\lambda, -\Delta^{\mathscr{D}}) \leqslant \lambda^{d/2} L_d^{cl} |\Omega|, \qquad \lambda > 0,$$

*or equivalently,*

$$\lambda_k \geqslant (L_d^{cl} |\Omega|)^{-2/d} k^{2/d}, \qquad k \in \mathbb{N}.$$

*Proof.* Let $-\Delta_j^{\mathscr{D}}$ be the Dirichlet Laplacian in $\Omega_j$, $j = 1, 2$. Since $\Omega = \Omega_1 \times \Omega_2$, the eigenvalues of $-\Delta^{\mathscr{D}}$ in $\Omega$ are equal to

$$\lambda_{lk} = \rho_l + \eta_k, \qquad l, k \in \mathbb{N},$$

where $\rho_l$ and $\eta_k$ are the eigenvalues of $-\Delta_1^{\mathscr{D}}$ and $-\Delta_2^{\mathscr{D}}$ respectively. Our assumptions on $-\Delta_1^{\mathscr{D}}$ imply

$$N(\rho, -\Delta_1^{\mathscr{D}}) \leqslant \rho^{d_1/2} L_{d_1}^{cl} |\Omega_1|.$$

Therefore

$$\begin{aligned} N(\lambda, -\Delta^{\mathscr{D}}) &= \#\{(l, k) \in \mathbb{N} \times \mathbb{N} : \rho_l + \eta_k < \lambda\} \\ &= \#\{(l, k) \in \mathbb{N} \times \mathbb{N} : \rho_l < (\lambda - \eta_k)_+\} \\ &\leqslant L_{d_1}^{cl} |\Omega_1| \sum_k (\lambda - \eta_k)_+^{d_1/2}. \end{aligned} \tag{2.9}$$

Let us first assume that $d_1 = 2$. Then by applying (2.5) to $-\Delta_2^{\mathscr{D}}$ we find

$$\begin{aligned} N(\lambda, -\Delta^{\mathscr{D}}) &\leqslant \lambda^{1 + d_2/2} L_2^{cl} |\Omega_1| (2\pi)^{-d_2} |\Omega_2| \int_{\mathbb{R}^{d_2}} (1 - |\xi|^2)_+ \, d\xi \\ &= \lambda^{1 + d_1/2} L_2^{cl} L_{d_2}^{cl} \frac{2}{(d_2 + 2)} |\Omega_1| \, |\Omega_2| = \lambda^{d/2} L_d^{cl} |\Omega|. \end{aligned}$$

Let

$$\mathscr{B}(p, q) = \int_0^1 v^{p-1}(1-v)^{q-1}\, dv = \frac{\Gamma(p)\, \Gamma(q)}{\Gamma(p+q)}$$

be the Beta function. If $d_1 > 2$, then using the same arguments we find

$$\begin{aligned}
&\sum_k (\lambda - \eta_k)_+^{d_1/2} \\
&\quad = \mathscr{B}(d_1/2 - 1, 2)^{-1} \sum_{k:\, \lambda > \eta_k} \int_0^\infty v^{d_1/2-2}(\lambda - \eta_k - v)_+\, dv \\
&\quad \leqslant \mathscr{B}(d_1/2 - 1, 2)^{-1} \sum_k \int_0^\infty v^{d_1/2-2}(\lambda - v - \eta_k)_+\, dv \\
&\quad \leqslant \mathscr{B}(d_1/2 - 1, 2)^{-1}\, |\Omega_2|\, \frac{2}{d_2+2}\, L_{d_2}^{cl} \int_0^\infty v^{d_1/2-2}(\lambda - v)_+^{d_2/2+1}\, dv \\
&\quad = \lambda^d\, |\Omega_2|\, \frac{2}{d_2+2}\, L_{d_2}^{cl}\, \mathscr{B}(d_1/2 - 1, 2)^{-1}\, \mathscr{B}(d_1/2 - 1, d_2/2 + 2). \qquad (2.10)
\end{aligned}$$

Collecting together all the constants in (2.9) and (2.10), we complete the proof. ▮

COROLLARY 2.9. *Under the conditions of Theorem* 2.8, *if* $\Omega_1 \subset \mathbb{R}^2$ *is a tiling domain, then the Pólya conjecture holds true.*

**4.** Let us consider the eigenvalue problem for the equations of classical elasticity

$$-a\Delta u_j - (a+b)\frac{\partial}{\partial x_j}(\nabla \cdot u) = \lambda u_j, \qquad (2.11)$$

$$u_j\,|_{\partial\Omega} = 0, \qquad j = 1, 2, 3, \qquad x \in \Omega \subset \mathbb{R}^3, \qquad (2.12)$$

where $a$ and $b$ denote the Lamé constants, $a, b > 0$ and $u = (u_1, u_2, u_3)$ is the elastic displacement vector. In this case $B(\xi)$ is equal to the matrix

$$B(\xi) = (a+b)\cdot \begin{pmatrix} \frac{a}{a+b}|\xi|^2 + \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \frac{a}{a+b}|\xi|^2 + \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \frac{a}{a+b}|\xi|^2 + \xi_3^2 \end{pmatrix},$$

$$\xi \in \mathbb{R}^3.$$

Its eigenvalues are

$$\nu_1 = a\,|\xi|^2, \qquad \nu_2 = a\,|\xi|^2, \qquad \text{and} \qquad \nu_3 = (2a+b)\,|\xi|^2.$$

Thus we obtain

$$\int_{\mathbb{R}^3} \operatorname{Tr} \varphi_1(B(\xi))\,d\xi = \frac{8\pi}{15}\,(2a^{-3/2} + (2a+b)^{-3/2}).$$

Applying Theorem 2.5 with $\alpha = 2$ we derive

COROLLARY 2.10. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$. If $B_{\mathscr{D}}$ is the operator of classical elasticity* (2.11), (2.12), *then*

$$N(\lambda, B_{\mathscr{D}}) \leqslant \lambda^{3/2}(2\pi^2)^{-1}\,5^{3/2}3^{-5/2}(2a^{-3/2} + (2a+b)^{-3/2})\,|\Omega|,$$

*or equivalently*

$$\lambda_k \geqslant \frac{3a}{5}\left(\frac{3}{2+(2+b/a)^{-3/2}}\right)^{2/3} \cdot \left(\frac{2\pi^2 k}{|\Omega|}\right)^{2/3}. \tag{2.13}$$

*Remark* 2.11. Formula (2.13) is an improvement of the inequality (1.19) obtained in [LP]. This became possible because the right hand side in (2.6) involves the trace $\operatorname{Tr} \varphi_1(B(\xi))$ rather than $m \cdot \max_{j=1,\dots,m} \nu_j(B(\xi))$, where $\nu_j(B(\xi))$ are the eigenvalues of the matrix $B(\xi)$.

## 3. NEUMANN BOUNDARY VALUE PROBLEM

**1.** Let us consider a differential operator

$$A(D)\,u(x) = \sum_{\beta \leqslant l} A_\beta D^\beta u(x), \qquad u \in C^\infty(\bar{\Omega}, \mathbb{C}^m), \qquad m \in \mathbb{N},$$

where $\Omega \subset \mathbb{R}^d$ is an open set and the coefficients $A_\beta$ are $m \times m$-matrices independent of $x \in \Omega$. Let us introduce a quadratic form

$$\mathfrak{B}_{\bar{\Omega}}[u, u] = \int_\Omega \|A(D)\,u\|^2\,dx, \qquad u \in C^\infty(\bar{\Omega}, \mathbb{C}^m),$$

where $\bar{\Omega}$ is the closure of the set $\Omega$. This form is semibounded from below. Let us study the completion of $C^\infty(\bar{\Omega}, \mathbb{C}^m)$ with respect to the quadratic form $\mathfrak{B}_{\bar{\Omega}}[u, u] + \int_{\bar{\Omega}} \|u\|^2 \, dx$ and let $B_{\mathcal{N}}$ be the corresponding Friedrichs extension of the differential operator $B(D) = A^*(D)\, A(D)$. The operator $B_{\mathcal{N}}$ can be naturally considered as an operator of the Neumann boundary problem in the domain $\Omega$ for the differential operator whose symbol is equal to $B(\xi) := A^*(\xi)\, A(\xi)$.

Let us assume that the spectrum of this operator is discrete and consists of $0 = \mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \cdots$, and that $\mu_k \to \infty$ as $k \to \infty$.

We put aside the problem of the discreteness of the spectrum of $B_{\mathcal{N}}$. For example, in the scalar case when $B(\xi) = |\xi|^{2l}$ the discreteness of the spectrum of this operator is equivalent to the compactness of the embedding $H^l(\Omega) \to L^2(\Omega)$. The latter requires some restrictive assumption on $\Omega$. The precise conditions of the compactness of this embedding are given in [Mz].

THEOREM 3.1. *Let $|\Omega| < \infty$ and assume that the spectrum of the operator $B_{\mathcal{N}}$ is discrete, $\mu_k \to \infty$ as $k \to \infty$. Then*

$$\sum_k (\mu - \mu_k)_+ \geqslant (2\pi)^{-d}\, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\mu(B(\xi))\, d\xi, \qquad \mu \geqslant 0. \tag{3.1}$$

*Proof.* Let $\omega_k$, be the orthonormal basis of eigenfunctions of the operator $B_{\mathcal{N}}$ whose respective eigenvalues are $\mu_k$, $k = 1, 2, ....$ Denote

$$e_\xi(x) = \begin{cases} \exp(ix\xi), & \text{as} \quad x \in \Omega, \\ 0, & \text{as} \quad x \notin \Omega, \end{cases}$$

and introduce the orthonormal basis $\{\tau_j(\xi)\}_{j=1}^m$ consisting of the eigenvectors of the matrix $B(\xi)$. Then

$$\begin{aligned} \sum_k (\mu - \mu_k)_+ &= \operatorname{Tr} \varphi_\mu(B_{\mathcal{N}}) \\ &= \sum_k \varphi_\mu(\mu_k) \int_\Omega \|\omega_k(x)\|^2 \, dx \\ &= (2\pi)^{-d} \sum_k \varphi_\mu(\mu_k) \int_{\mathbb{R}^d} \|\hat{\omega}_k(\xi)\|^2 \, d\xi \\ &= (2\pi)^{-d} \sum_k \sum_{j=1}^m \varphi_\mu(\mu_k) \int_{\mathbb{R}^d} |(\hat{\omega}_k(\xi), \tau_j(\xi))|^2 \, d\xi. \end{aligned} \tag{3.2}$$

Let $E_\nu$, $\nu \in \mathbb{R}$, be the spectral projection of the selfadjoint operator $B_{\mathcal{N}}$. We can now rewrite (3.2) as

$$\operatorname{Tr} \varphi_\mu(B_{\mathcal{N}}) = \int_{\mathscr{R}^d} \sum_k \varphi_\mu(\mu_k) \sum_{j=1}^m \int_\Omega \int_\Omega (\omega_k(x), \tau_j(\xi)\, e_\xi(x)) \times (\tau_j(\xi)\, e_\xi(y), \omega_k(y))\, dy\, dx\, d\xi$$
$$= \sum_{j=1}^m \int_{\mathbb{R}^d} \int_0^\infty \varphi_\mu(v)(dE_v e_\xi \tau_j(\xi), e_\xi \tau_j(\xi))\, d\xi. \tag{3.3}$$

Since

$$|\Omega|^{-1} \int_0^\infty (dE_v e_\xi \tau_j, e_\xi \tau_j) = 1, \qquad \forall \xi \in \mathbb{R}^d, \qquad j = 1, 2, ..., m,$$

then by applying the Jensen inequality to the right hand side of (3.3) we obtain

$$\operatorname{Tr} \varphi_\mu(B_{\mathcal{N}}) \geqslant |\Omega| \int_{\mathbb{R}^d} \sum_{j=1}^m \varphi_\mu \left( \frac{1}{|\Omega|} \int_0^\infty v(dE_v e_\xi \tau_j(\xi), e_\xi \tau_j(\xi)) \right) d\xi.$$

Notice that

$$\int_0^\infty v(dE_v e_\xi \tau_j, e_\xi \tau_j) = \mathfrak{B}_{\bar{\Omega}}[e_\xi \tau_j(\xi), e_\xi \tau_j(\xi)]$$
$$= \int_\Omega \|A(D)\, e_\xi \tau_j(\xi)\|^2\, dx$$
$$= |\Omega|\, \|A(\xi)\, \tau_j(\xi)\|^2$$
$$= |\Omega| (B(\xi)\, \tau_j(\xi), \tau_j(\xi)).$$

Since $\tau_j(\xi)$ are the eigenvectors of the matrix-function $B(\xi)$, we have

$$\sum_{j=1}^m \varphi_\mu \left( \frac{1}{|\Omega|} \int_0^\infty v(dE_v e_\xi \tau_j(\xi), e_\xi \tau_j(\xi)) \right) = \operatorname{Tr} \varphi_\mu(B(\xi)).$$

This leads to (3.1) and completes the proof. ∎

**2.** Apply now the inequality (3.1) to the counting function of the spectrum of the operator $B_{\mathcal{N}}$.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$ and $B(\xi)$ be a positively homogeneous symbol of degree $2l$. Then under the assumptions of Theorem* 3.1 *we have*

$$N(\mu, B_{\mathcal{N}}) \geqslant \mu^{d/2l} (2\pi)^{-d}\, |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi))\, d\xi, \qquad \mu \geqslant 0. \tag{3.4}$$

*Proof.* Since the first eigenvalue of the operator $B_{\mathcal{N}}$ is equal to zero we obtain that

$$N(\mu, B_{\mathcal{N}}) \geqslant \frac{1}{\mu} \sum_k (\mu - \mu_k)_+, \qquad \mu \geqslant 0.$$

Theorem 3.1 and the homogeneity of the matrix $B(\xi)$ lead us to

$$N(\mu, B_{\mathcal{N}}) \geqslant \frac{1}{\mu} (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_\mu(B(\xi))\, d\xi$$

$$= \mu^{d/2l} (2\pi)^{-d} |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr} \varphi_1(B(\xi))\, d\xi. \qquad \blacksquare$$

Let $m = 1$ and $B(\xi) = |\xi|^{2l}$, $l \in \mathbb{N}$. Then the operator $B_{\mathcal{N}}$ coincides with the operator of the Neumann boundary problem for $(-\Delta)^l$. In this case

$$(2\pi)^{-d} \int_{\mathbb{R}^d} \varphi_1(B(\xi))\, d\xi = L_d^{cl} \frac{2l}{(d+2l)}.$$

and (3.4) implies.

COROLLARY 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set of finite measure, $|\Omega| < \infty$. Then*

$$N(\mu, ((-\Delta)^l)_{\mathcal{N}}) \geqslant \mu^{d/2l} L_d^{cl} \frac{2l}{d+2l} |\Omega|, \qquad \mu \geqslant 0. \tag{3.5}$$

*Remark* 3.4. If $l = 1$, then (3.5) is equivalent to the inequality (1.3) proved in [K1].

## 4. MORE EIGENVALUE ESTIMATES FOR THE DIRICHLET LAPLACIAN

Let $B_{\mathcal{D}} = -\Delta^{\mathcal{D}}$ in $L^2(\Omega)$, $\Omega \subset \mathbb{R}^d$ and let us assume that the spectrum of this operator is discrete. Let $\omega_1, \omega_2, ...$ be the orthonormal basis of eigenfunctions of the operator $B_{\mathcal{D}}$ whose respective eigenvalues are $0 < \lambda_1 < \lambda_2 \leqslant \cdots$. Denote

$$\tilde{\omega} = \sup_{x \in \Omega} |\omega_1(x)|. \tag{4.1}$$

Using an argument similar to those we used in Section 3 we can prove the following statement.

THEOREM 4.1. *Let the spectrum of the operator $-\Delta^{\mathcal{D}}$ in $L^2(\Omega)$ be discrete. Then for any $\lambda > 0$ we have*

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) = \sum_k (\lambda - \lambda_k)_+ \geqslant (\lambda - \lambda_1)_+^{1+d/2} L_d^{cl} \frac{2}{d+2} \tilde{\omega}^{-2}. \tag{4.2}$$

*Proof.* The functions

$$\theta_\xi(x) := \omega_1 e^{-i(x,\xi)}, \qquad \xi \in \mathbb{R}^d,$$

belong to the domain of the operator $-\Delta^{\mathcal{D}}$. Obviously

$$\begin{aligned}\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) &= \sum_k \varphi_\lambda(\lambda_k) \int |\omega_k|^2 \, dx \\ &\geqslant \tilde{\omega}^{-2} \sum_k \varphi_\lambda(\lambda_k) \int |\omega_1 \omega_k|^2 \, dx \\ &= (2\pi)^{-d} \tilde{\omega}^{-2} \sum_k \varphi_\lambda(\lambda_k) \int \left| \int \omega_k \theta_\xi(x) \, dx \right|^2 d\xi.\end{aligned}$$

If the spectral projection of the operator $-\Delta^{\mathcal{D}}$ is denoted by $E_\nu$, then the last expression can be rewritten as

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) \geqslant (2\pi)^{-d} \tilde{\omega}^{-2} \int_{\mathbb{R}^d} \int_0^\infty \varphi_\lambda(\nu)(dE_\nu \theta_\xi, \theta_\xi) \, d\xi.$$

Clearly

$$\int_0^\infty (dE_\nu \theta_\xi, \theta_\xi) = \|\theta_\xi\|^2_{L^2(\Omega)} = \|\omega_1\|^2_{L^2(\Omega)} = 1,$$

and by using the Jensen inequality we obtain

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) \geqslant (2\pi)^{-d} \tilde{\omega}^{-2} \int \varphi_\lambda \left( \int_0^\infty \nu (dE_\nu \theta_\xi, \theta_\xi) \right) d\xi. \tag{4.3}$$

A simple calculation gives

$$\int_0^\infty \nu (dE_n \theta_\xi, \theta_\xi) = \int_{\mathbb{R}^d} |\nabla \theta_\xi|^2 \, dx = (|\xi|^2 + \lambda_1). \tag{4.4}$$

Combining (4.4) in (4.3) we arrive at

$$\operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}}) \geqslant (2\pi)^{-d} \tilde{\omega}^{-2} \int_{\mathbb{R}^d} \varphi_\lambda(|\xi|^2 + \lambda_1)\, d\xi$$
$$= (\lambda - \lambda_1)_+^{1+d/2} L_d^{cl} \frac{2}{d+2} \tilde{\omega}^{-2}.$$

The theorem is proved. ▮

In particular, if $\lambda = \lambda_2$ in (4.2), then we obtain the following upper estimate for the difference of the two first eigenvalues for the Dirichlet Laplacian.

COROLLARY 4.2. *Under the conditions of Theorem* 4.1 *we have*

$$\lambda_2 - \lambda_1 \leqslant \left( L_d^{cl} \frac{2}{d+2} \right)^{-2/d} \tilde{\omega}^{4/d}.$$

*Remark* 4.3. Some other upper estimates on $\lambda_2 - \lambda_1$ were studied in [PPW] and [SWYY] (see also [SY]).

If $\lambda > \lambda_1$, then

$$N(\lambda, -\Delta^{\mathcal{D}}) \geqslant \frac{1}{\lambda - \lambda_1} \operatorname{Tr} \varphi_\lambda(-\Delta^{\mathcal{D}})$$

and by using (4.2) we derive

COROLLARY 4.4. *If the conditions of Theorem* 4.1 *are satisfied, then for any* $\lambda > \lambda_1$ *we obtain*

$$N(\lambda, -\Delta^{\mathcal{D}}) \geqslant (\lambda - \lambda_1)^{d/2} L_d^{cl} \frac{2}{d+2} \tilde{\omega}^{-2}.$$

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